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CONCOMITANT BINARY FORMS IN TERMS OF THE ROOTS.*

By Miss Annie L. MacKinnon, Ithaca, N. Y.

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* Dissertation presented to the Faculty of Cornell University in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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PREFACE.

In the preface to Gordan's Lectures on the Invariant Theory,* Dr. Kerschensteiner speaks of three books as containing the substance of the modern Invariant Theory—the books of Salmon, Clebsch, and Faá de Bruno. Adding to these Gordan's Lectures and Burnside and Panton's Theory of Equations, we include the principal works which give a general presentation of the Invariant Theory of the present time. In these five books, the prominence given to the expression of Covariants and Invariants in terms of the roots is various. Faá de Bruno's treatment of the root expressions is the most extensive; he gives tables of the Invariants (not Covariants) of the lower binary quantics through the Sextic (omitting B and D of the Sextic), expressed as functions of root differences. There is no suggestion of any system for calculating the root expressions of these tables, aside from the use of coefficients expressed as symmetric functions of the roots.

Burnside and Panton approach the subject of Covariants and Invariants through the expressions of symmetric functions of root differences, and make use of symmetric functions of the roots to establish connections between the root and coefficient forms.

Scattered through Salmon's book on Modern Higher Algebra are many of the simpler Covariant and Invariant root expressions. The methods presented by Salmon for the calculation of root forms are based upon symmetric functions of the roots, symmetric functions of root differences, and upon the use of any convenient geometric relation obtained through the coefficient forms, and also upon the use of transformed equations. There is in this book, no recognition of symbolic root forms nor of the possibility of the application of Cayley's symbolic operators to the calculation of root expressions for Covariants and Invariants.

Both Clebsch and Gordan touch upon a theory of symbolic root forms, the theory to be presented in this paper. Clebsch appears not to have recognized, or if he recognized has not made clear, the directness and simplicity of the connection which exists between root and coefficient symbolic expressions for Covariants and Invariants. Gordan fully recognizes the relation between the two forms of symbolic expression; and the work which follows in this paper, though developed independently of Gordan's work in this line, is in reality an application of the underlying principle of the Gordan symbolism.

* Gordan's Vorlesungen über Invariantentheorie—Kerschensteiner, Leipzig, 1885.

As far as I have been able to ascertain there has been in English writings no recognition of symbolic methods applied to the expression of Covariants and Invariants in terms of the roots, excepting (possibly) the following sentence by Sylvester:* "Gordan's and Jordan's results concerning symbolic determinants are correlative and coextensive with theorems concerning root differences, so that the method of differentiants when fully developed would lead to the substitution of actual differences or determinants for symbolic determinants in the Gordan theory."

A realization of the substantial identity of the form of a Covariant root symbol with the form of its expression in the root differences, and of the directness of the interpretation of one form from the other, brings into clearer light the practical value of German Symbolism in Modern Algebra.

The following pages present the results of a study of root forms and of an attempt to systematize the calculation and comparison of Covariants and Invariants in terms of the roots. The subject is presented according to the following arrangement:

Part I.—General Theory.

1. Theory of Covariants and Invariants of binary quantics in terms of the roots.
2. Comparison of Root and Coefficient Symbols.
3. Tables of Covariants and Invariants of the lower quantics including the sextic, and of pairs of the first five quantics (including linear quantics).
4. Particular classes of Forms and Operators.

Part II.—Some Geometrical interpretations and applications.

1. Geometry of Binary Forms.
2. Particular Covariants and Invariants with geometrical interpretations.
3. Binary root forms in their relations to certain ternary forms.

* American Journal of Mathematics, Vol. II (1879), p. 329.

PART I.—GENERAL THEORY.

CHAPTER I.

THEORY OF COVARIANTS AND INVARIANTS IN TERMS OF THE ROOTS.

1. *Covariant and invariant defined.* A covariant of a binary quantic may be defined as a function of both the roots and variables which is altered only by a constant factor when the quantic is linearly transformed. It will be shown in the next article that this property is possessed by a function that is the product of three factors: (1) of a symmetric function of the differences of the roots and differences between the variable x/y and any roots into which each root enters the same number of times; (2) of the coefficient of the highest power of x in the quantic raised to a power equal to the number of times any one root appears in (1); (3) of y^m, m being the number of times x/y appears in (1).*

An invariant of a binary quantic is a function of the roots, but not of the variables, which is altered only by a constant factor when the quantic is linearly transformed. An invariant function is the product of two factors, one of which is a symmetric function of the differences of the roots, in which each root appears the same number of times, and the other the coefficient of the highest power of x in the quantic, raised to a power equal to the number of times any one root appears in the symmetric function. An invariant is a particular case of a covariant in which $m = 0$.

2. *Form of the general covariant.* Let the binary quantic

$$(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)\ldots \tag{1}$$

be represented by

$$\alpha\beta\gamma\ldots, \tag{2}$$

in which

$$\left.\begin{aligned} \alpha &= \alpha_1 x + \alpha_2 y\,, \\ \beta &= \beta_1 x + \beta_2 y\,, \\ &\ldots\ldots\ldots \end{aligned}\right\} \tag{3}$$

Equating quantic (1) to zero, it may be written

$$\left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right]\left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right]\left[\frac{x}{y} + \frac{\gamma_2}{\gamma_1}\right]\ldots = 0\,, \tag{4}$$

the roots of this equation in x/y being

$$-\frac{\alpha_2}{\alpha_1}, \quad -\frac{\beta_2}{\beta_1}, \quad -\frac{\gamma_2}{\gamma_1}, \quad \ldots. \tag{5}$$

*Salmon's Modern Higher Algebra, pp. 124–5. Burnside and Panton's Theory of Equations, Third Edition, pp. 367, 374.

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Again let θ be a function composed of the three factors described in Art. 1, such that

$$\theta \equiv (\alpha_1\beta_1\gamma_1\ldots\nu_1)^{\omega}y^{m}$$
$$\times\ \Sigma\left[\frac{\beta_2}{\beta_1}-\frac{\alpha_2}{\alpha_1}\right]^{e_{12}}\left[\frac{\gamma_2}{\gamma_1}-\frac{\beta_2}{\beta_1}\right]^{e_{23}}\cdots\left[\frac{\nu_2}{\nu_1}-\frac{\mu_2}{\mu_1}\right]^{e_{n-1,n}}\left[\frac{x}{y}+\frac{\alpha_2}{\alpha_1}\right]^{e_1}\cdots\left[\frac{x}{y}+\frac{\nu_2}{\nu_1}\right]^{e_n}, \quad (6)$$

wherein the e's are subject to the conditions

$$\left.\begin{array}{l} e_1 + e_{12} + e_{13} + \ldots + e_{1n} = \omega, \\ e_2 + e_{21} + e_{23} + \ldots + e_{2n} = \omega, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ e_n + e_{n1} + e_{n2} + \ldots + e_{nn} = \omega, \\ e_1 + e_2 + e_3 + \ldots + e_n = m, \end{array}\right\} \quad (7)$$

the symbol e_{rs} or e_{sr} denoting the exponent of the factor in which both the rth and sth of the letters $\alpha, \beta, \gamma \ldots$ occur, and e_r being the exponent of the linear factor that involves the rth letter.

Equation (6) may also be put in the form

$$\theta = \Sigma(\alpha_1\beta_2-\alpha_2\beta_1)^{e_{12}}(\beta_1\gamma_2-\beta_2\gamma_1)^{e_{23}}\ldots(\mu_1\nu_2-\mu_2\nu_1)^{e_{n-1,n}}(\alpha_1x+\alpha_2y)^{e_1}\ldots(\nu_1x+\nu_2y)^{e_n}, \quad (8)$$

which will be briefly written

$$\theta = \Sigma(\alpha\beta)^{e_{12}}(\beta\gamma)^{e_{23}}\ldots(\mu\nu)^{e_{n-1,n}}\,\alpha^{e_1}\ldots\nu^{e_n}, \quad (9)$$

wherein

$$(\alpha\beta) = (\alpha_1\beta_2-\alpha_2\beta_1),\ \ldots,\ \alpha = \alpha_1x+\alpha_2y,\ \ldots, \quad (10)$$

and in which it will be noticed the letters $\alpha, \beta, \gamma, \ldots$ each enter the same number of times.

To show that θ conforms to the definition in Art. 1, let x and y be transformed by the general linear substitution*

$$y = \lambda\bar{y} - \mu\bar{x}, \quad x = -\lambda'\bar{y} + \mu'\bar{x}, \quad (11)$$

then the root quantities $\alpha_1, \alpha_2, \beta_1, \beta_2, \ldots$ are transformed by the substitutions

$$\left.\begin{array}{l} \alpha_1 = \lambda\bar{\alpha}_1 + \mu\bar{\alpha}_2, \quad \alpha_2 = \lambda'\bar{\alpha}_1 + \mu'\bar{\alpha}_2, \\ \beta_1 = \lambda\bar{\beta}_1 + \mu\bar{\beta}_2, \quad \beta_2 = \lambda'\bar{\beta}_1 + \mu'\bar{\beta}_2, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}\right\} \quad (12)$$

---

* The arrangement in (11) is adopted to secure symmetry in (12).

which are obtained from (11) by changing x into $-\alpha_2, -\beta_2, \ldots$ and y into $\alpha_1, \beta_1, \ldots$ in accordance with (5),

$$\left.\begin{aligned} \therefore\quad \alpha_1\beta_2 - \alpha_2\beta_1 &= (\lambda\mu' - \lambda'\mu)(\bar{\alpha}_1\bar{\beta}_2 - \bar{\alpha}_2\bar{\beta}_1)\,, \\ \beta_1\gamma_2 - \beta_2\gamma_1 &= (\lambda\mu' - \lambda'\mu)(\bar{\beta}_1\bar{\gamma}_2 - \bar{\beta}_2\bar{\gamma}_1)\,, \\ &\ldots\ldots\ldots\ldots \\ \alpha_1 x + \alpha_2 y &= (\lambda\mu' - \lambda'\mu)(\bar{\alpha}_1\bar{x} + \bar{\alpha}_2\bar{y})\,, \\ \beta_1 x + \beta_2 y &= (\lambda\mu' - \lambda'\mu)(\bar{\beta}_1\bar{x} + \bar{\beta}_2\bar{y})\,, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{13}$$

and (8) transforms into

$$\Theta = (\lambda\mu' - \lambda'\mu)^{\frac{1}{2}(n\omega+m)}$$
$$\times\, \Sigma(\bar{\alpha}_1\bar{\beta}_2 - \bar{\alpha}_2\bar{\beta}_1)^{e_{12}}(\bar{\beta}_1\bar{\gamma}_2 - \bar{\beta}_2\bar{\gamma}_1)^{e_{23}}\ldots(\bar{\alpha}_1\bar{x} + \bar{\alpha}_2\bar{y})^{e_1}(\bar{\beta}_1\bar{x} + \bar{\beta}_2\bar{y})^{e_2}\ldots \tag{14}$$

which may also be written in the original form, as in (6),*

$$\Theta = (\lambda\mu' - \lambda'\mu)^{\frac{1}{2}(n\omega+m)}(\bar{\alpha}_1\bar{\beta}_1\bar{\gamma}_1\ldots\bar{\nu}_1)^{\omega}\,\bar{y}^{m}$$
$$\times\, \Sigma\left[\frac{\bar{\beta}_2}{\bar{\beta}_1} - \frac{\bar{\alpha}_2}{\bar{\alpha}_1}\right]^{e_{12}}\left[\frac{\bar{\gamma}_2}{\bar{\gamma}_1} - \frac{\bar{\beta}_2}{\bar{\beta}_1}\right]^{e_{23}}\ldots\left[\frac{\bar{x}}{\bar{y}} + \frac{\bar{\alpha}_2}{\bar{\alpha}_1}\right]^{e_1}\ldots \tag{15}$$

$$\therefore\qquad \Theta = (\lambda\mu' - \lambda'\mu)^{\frac{1}{2}(n\omega+m)}\,\bar{\Theta}\,.$$

Hence Θ and $\bar{\Theta}$ differ only by a constant factor which is a power of the modulus of transformation, and Θ is a covariant.

Of the two forms for Θ given in (6) and (8) the latter (or rather its abbreviation in (9)) will be chiefly used. E. g. the H and J covariants of a cubic are

$$\Sigma(\alpha\beta)^2\gamma^2\,,\quad \Sigma(\alpha\beta)^2(\alpha\gamma)\,\beta\gamma^2\,.$$

3. *Form of the general invariant.* The function Φ is an invariant when

$$\Phi = (\alpha_1\beta_1\gamma_1\ldots)^{\omega}\,\Sigma\left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{12}}\left[\frac{\gamma_2}{\gamma_1} - \frac{\beta_2}{\beta_1}\right]^{e_{23}}\ldots\left[\frac{\nu_2}{\nu_1} - \frac{\mu_2}{\mu_1}\right]^{e_{n-1,n}}, \tag{1}$$

* The part played by the factors $(\alpha_1\beta_1\gamma_1\ldots)^{\omega}\,y^m$ in the original form is now evident, as without them Θ would not differ from $\bar{\Theta}$ merely by a power of the modulus. These factors become $a_0{}^{\omega}$ in covariants for the quantic form $a_0(x-\alpha)(x-\beta)(x-\gamma)\ldots$

and

$$\left.\begin{array}{l} e_{12} + e_{13} + e_{14} + \ldots + e_{1n} \quad = \omega \,, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ e_{1n} + e_{2n} + e_{3n} + \ldots + e_{n-1,n} = \omega \,. \end{array}\right\} \qquad (2)$$

As stated in Art. 1, Φ is a covariant in which m equals zero. Equation (1) will be written in the symbolic form

$$\Phi = \Sigma(\alpha\beta)^{e_{12}}(\beta\gamma)^{e_{23}} \ldots (\mu\nu)^{e_{n-1,n}} \qquad (3)$$

where $(\alpha\beta) = (\alpha_1\beta_2 - \alpha_2\beta_1), \ldots$.

Examples of invariants in the form (3) are $(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^2$ which is D of the cubic, and $\Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\gamma)(\beta\delta)$ which is T of the quartic.

4. *Order, weight, and degree of covariants and invariants.** The order of a covariant or invariant is the number of times any one root appears in any term of its root expression†—it is the ω of Arts. 2, 3; its *degree* is the number of times x appears in any term and is equal to the m of Art. 2.

The *weight* of a covariant or invariant is the number of determinant factors in its symmetric function, and is the sum of all exponents with a double subscript in (9), Art. 2, or (3), Art. 3.

Let n = degree of a quantic,
m = degree of a covariant of the quantic,
ω = order of a covariant,
κ = weight of a covariant.

Now, equations (7), Art. 2, form a set of n equations in which each term e_{rs} occurs twice; and in which, therefore, the sum of the terms with double subscripts is 2κ; while the sum of the terms $e_1, e_2, \ldots, e_n$, each of which occurs but once, is m. Adding, this set of n equations gives

$$2\kappa + m = n\omega\,, \qquad (1)$$

$$\therefore \quad m = n\omega - 2\kappa\,, \qquad (2)$$

which is true for any covariant.‡ An invariant is a covariant in which $m = 0$; therefore, in any invariant

$$n\omega = 2\kappa\,; \qquad (3)$$

hence the product $n\omega$ must be an even number.

* Salmon's usage of these terms is followed.

† The terms of the root expression are understood to be the terms in the symmetric root function; thus $(\alpha\beta)^2\gamma^2\delta^2$ is one term in $\Sigma(\alpha\beta)^2\gamma^2\delta^2$, and $(\alpha\gamma)^2\beta^2\delta^2$ is another term.

‡ See Burnside and Panton, Theory of Equations, p. 370. Salmon, M. H. Algebra, p. 130.

5. *Reduction formulæ.* From the three equations

$$\alpha = \alpha_1 x + \alpha_2 y\,, \quad \beta = \beta_1 x + \beta_2 y\,, \quad \gamma = \gamma_1 x + \gamma_2 y\,, \tag{1}$$

there arises the determinant

$$\begin{vmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \beta_1 & \beta_2 \\ \gamma & \gamma_1 & \gamma_2 \end{vmatrix} = 0\,, \tag{2}$$

or

$$\alpha(\beta\gamma) + \beta(\gamma\alpha) + \gamma(\alpha\beta) = 0\,; \tag{3}$$

and from the four equations

$$\alpha = \alpha_1 x + \alpha_2 y\,, \quad \beta = \beta_1 x + \beta_2 y\,, \quad \gamma = \gamma_1 x + \gamma_2 y\,, \quad \delta = \delta_1 x + \delta_2 y\,, \tag{4}$$

is obtained

$$(\alpha\delta)(\beta\gamma) + (\beta\delta)(\gamma\alpha) + (\gamma\delta)(\alpha\beta) = 0\,. \tag{5}$$

By means of formulas (3) and (5), and others derived from them, the general symbolic forms Φ, Θ, can be expressed in terms of irreducible* covariants and invariants. These reductions are similar to those of Clebsch and Gordan.†

E. g. to reduce $\Sigma(\alpha\beta)(\gamma\alpha)\beta\gamma$. From formula (3)

$$2(\alpha\beta)(\gamma\alpha)\beta\gamma = \alpha^2(\beta\gamma)^2 - \beta^2(\gamma\alpha)^2 - \gamma^2(\alpha\beta)^2\,, \tag{6}$$

$$\therefore \quad 2\Sigma(\alpha\beta)(\gamma\alpha)\beta\gamma = \Sigma\{\alpha^2(\beta\gamma)^2 - \beta^2(\gamma\alpha)^2 - \gamma^2(\alpha\beta)^2\} \tag{7}$$

$$\therefore \quad 2\Sigma(\alpha\beta)(\gamma\alpha)\beta\gamma = -\Sigma(\alpha\beta)^2\gamma^2\,, \tag{8}$$

i. e. $\Sigma(\alpha\beta)(\gamma\alpha)\beta\gamma$ is a multiple of $\Sigma(\alpha\beta)^2\gamma^2$.

Similarly $\Sigma(\beta\gamma)^2(\alpha\beta)(\gamma\alpha)\alpha^2\beta\gamma$ is a multiple of $\Sigma(\beta\gamma)^2(\alpha\beta)^2\alpha^2\gamma^2$.

6. *Transvection.* The operation $\left\{\dfrac{d}{dx}\cdot\left[\dfrac{d}{dy}\right] - \dfrac{d}{dy}\cdot\left[\dfrac{d}{dx}\right]\right\}^k\varphi\psi$, in which the differential operators in parentheses act upon ψ, and the others upon φ, will be represented by $\{\varphi\psi\}^k$; it is called "the kth transvectant of φ over ψ."

Let

$$\varphi = \alpha = \alpha_1 x + \alpha_2 y\,, \tag{1}$$

$$\psi = \beta = \beta_1 x + \beta_2 y\,, \tag{2}$$

then

$$\{\varphi\psi\} = \frac{d\alpha}{dx}\cdot\frac{d\beta}{dy} - \frac{d\alpha}{dy}\cdot\frac{d\beta}{dx} = \alpha_1\beta_2 - \alpha_2\beta_1 = (\alpha\beta)\,, \tag{3}$$

---

* Salmon, M. H. Algebra, p. 175.

† See Clebsch and Gordan; also Salmon, p. 318.

and the determinant symbol $(\alpha_i\beta)$ may be considered the 1st transvectant of α over $_i\beta$. All invariants are made up of the transvectants $(\alpha_i\beta)$, $(_i\beta\gamma)$, etc. (Art. 2); and all covariants are made up of these transvectants and the linear quantic factors. For convenience α, β, γ, ... will be called quantic factors, and $(\alpha_i\beta)$, $(_i\beta\gamma)$, ... invariant factors; while the name transvectant will be reserved for the general operation $\{\varphi\psi\}^k$.

7. *The first transvectant.* Any two covariants of a quantic, expressed as rational, integral functions of quantic and invariant factors, are of forms like

$$\varphi = A_{1i}\beta\gamma\delta \ldots + A_2\alpha\gamma\delta \ldots + A_3\alpha_i\beta\varepsilon \ldots + \ldots, \tag{1}$$

$$\psi = B_1\alpha^2{}_i\beta\gamma \ldots + B_2\alpha_i\beta^2\gamma \ldots + B_3\alpha_i\beta\gamma^2 \ldots + \ldots; \tag{2}$$

wherein A_r = the invariant factors of the rth term of φ,

B_r = the invariant factors of the rth term of ψ.

For these two forms the first transvectant is

$$\{\varphi\psi\} = \left\{\frac{d\varphi}{dx}\cdot\frac{d\psi}{dy} - \frac{d\varphi}{dy}\cdot\frac{d\psi}{dx}\right\}$$

$$= A_1B_1\{(_i\beta\alpha)\gamma\delta \ldots \alpha_i\beta\gamma \ldots + (_i\beta\gamma)\gamma\delta \ldots \alpha^2{}_i\beta\varepsilon \ldots + \ldots\}$$

$$+ A_1B_2\{(_i\beta\alpha)\gamma\delta \ldots {}_i\beta^2\gamma \ldots + (_i\beta\gamma)\gamma\delta \ldots \alpha_i\beta^2\varepsilon \ldots + \ldots\} + \ldots \tag{3}$$

$$= (\alpha_i\beta)\left\{\frac{d\varphi}{d\alpha}\cdot\frac{d\psi}{d_i\beta} - \frac{d\varphi}{d_i\beta}\cdot\frac{d\psi}{d\alpha}\right\} + (_i\beta\gamma)\left\{\frac{d\varphi}{d_i\beta}\cdot\frac{d\psi}{d\gamma} - \frac{d\varphi}{d\gamma}\cdot\frac{d\psi}{d_i\beta}\right\} + \ldots. \tag{4}$$

In the result (3) of this transvection, each term contains a new invariant factor and the invariant factors of one term of φ and one term of ψ; and in each term any letter appears the same number of times that it appeared in a term of φ and a term of ψ taken together; and therefore the form (3) is a covariant. The letters α, β, γ ... have changed their places by pairs from quantic to invariant factors,* one pair in each term, and one member of each pair from φ and the other from ψ.

8. *The rth transvectant.* Let φ be any covariant. Then

$$\left.\begin{aligned} \frac{d\varphi}{dx} &= \left[\alpha_1\frac{d}{d\alpha} + \beta_1\frac{d}{d_i\beta} + \ldots\right]\varphi, \\ \frac{d^2\varphi}{dx^2} &= \left[\alpha_1\frac{d}{d\alpha} + \beta_1\frac{d}{d_i\beta} + \ldots\right]^2\varphi, \\ &\ldots\ldots\ldots\ldots\ldots \\ \frac{d^r\varphi}{dx^r} &= \left[\alpha_1\frac{d}{d\alpha} + \beta_1\frac{d}{d_i\beta} + \ldots\right]^r\varphi; \end{aligned}\right\} \tag{1}$$

* Gordan calls this change of letter by pairs from quantic to invariant factors, the *Faltung Process*. See Gordan's Vorlesungen über Invariantentheorie, p. 10.

$$\left.\begin{aligned}\frac{d\varphi}{dy} &= \left[a_2\frac{d}{da} + \beta_2\frac{d}{d\beta} + \dots\right]\varphi\,, \\ \frac{d^2\varphi}{dy^2} &= \left[a_2\frac{d}{da} + \beta_2\frac{d}{d\beta} + \dots\right]^2\varphi\,, \\ &\dots\dots\dots\dots \\ \frac{d^r\varphi}{dy^r} &= \left[a_2\frac{d}{da} + \beta_2\frac{d}{d\beta} + \dots\right]^r\varphi\,;\end{aligned}\right\} \tag{2}$$

$$\frac{d^{r+s}\varphi}{dy^r dx^s} = \left[a_2\frac{d}{da} + \beta_2\frac{d}{d\beta} + \dots\right]^s \cdot \left[a_1\frac{d}{da} + \beta_1\frac{d}{d\beta} + \dots\right]^r\varphi\,. \tag{3}$$

Let $\xi_1 = \frac{d}{dx}$, $\xi_2 = \left[\frac{d}{dx}\right]$, $\eta_1 = \frac{d}{dy}$, $\eta_2 = \left[\frac{d}{dy}\right]$; then, if φ and ψ are any two covariants of an $n$ic,

$$\{\varphi\psi\}^2 = (\xi_1\eta_2 - \xi_2\eta_1)^2\,\varphi\psi = (\xi_1^2\,.\,\eta_2^2 - 2\xi_1\eta_1\,.\,\xi_2\eta_2 + \xi_2^2\,.\,\eta_1^2)\,\varphi\psi$$

$$= \left[a_1\frac{d}{da} + \dots\right]^2\varphi\,.\left[a_2\frac{d}{da} + \dots\right]^2\psi$$

$$- 2\left[a_1\frac{d}{da} + \dots\right]\left[a_2\frac{d}{da} + \dots\right]\varphi\,.\left[a_1\frac{d}{da} + \dots\right]\left[a_2\frac{d}{da} + \dots\right]\psi$$

$$+ \left[a_2\frac{d}{da} + \dots\right]^2\varphi\,.\left[a_1\frac{d}{da} + \dots\right]^2\psi$$

$$= (a\beta)^2\left\{\frac{d}{da}\left[\frac{d}{d\beta}\right] - \frac{d}{d\beta}\left[\frac{d}{da}\right]\right\}^2\varphi\psi$$

$$+ (\gamma a)^2\left\{\frac{d}{d\gamma}\left[\frac{d}{da}\right] - \frac{d}{da}\left[\frac{d}{d\gamma}\right]\right\}^2\varphi\psi + \dots$$

$$+ 2\,(a\beta)(\gamma a)\left\{\frac{d}{da}\left[\frac{d}{d\beta}\right] - \frac{d}{d\beta}\left[\frac{d}{da}\right]\right\}\left\{\frac{d}{d\gamma}\left[\frac{d}{da}\right] - \frac{d}{da}\left[\frac{d}{d\gamma}\right]\right\}\varphi\psi$$

$$+ \dots\,. \tag{4}$$

Similarly, for the rth transvectant

$$\{\varphi\psi\}^r = \Bigg[(a\beta)\left\{\frac{d}{da}\left[\frac{d}{d\beta}\right] - \frac{d}{d\beta}\left[\frac{d}{da}\right]\right\}$$

$$+ (\gamma a)\left\{\frac{d}{d\gamma}\left[\frac{d}{da}\right] - \frac{d}{da}\left[\frac{d}{d\gamma}\right]\right\} + \dots\Bigg]^r\varphi\psi\,. \tag{5}$$

From (4) and (5) it appears, that in the rth transvectant of φ over ψ, the letters $\alpha, \beta, \gamma, \ldots$ have changed their places by pairs from quantic to invariant factors, r pairs at a time, one member of each pair from φ and the other from ψ; and that each letter appears the same number of times in any term. The rth transvectant of φ over ψ is a covariant or invariant; its weight is $\varkappa + \varkappa' + r$, its degree is $m + m' - 2r$, and its order is $\omega + \omega'$, where the primed letters refer to ψ and the unprimed to φ. (See Art. 4.)

9. *Covariants and invariants as transvectants.* Since all covariants and invariants are combinations of quantic factors, and of the invariant factors which are derived by transvection from quantic factors, it is evident that all invariants and covariants are the result of the transvection of some covariant over some other covariant or the same covariant.* If f represent any binary quantic $\alpha\beta\gamma \ldots$, every possible covariant and invariant of f will be included among all the possible transvectants of f over f, and all the possible transvectants of these ff' transvectants, etc.

To illustrate the connection between the possible transvectants and the covariants and invariants of a quantic, we give the following table of possible transvectants of the cubic.†

$$
\begin{array}{lll}
\{ff'\} = 0\,, & \{ff'\}^2 = m_1 H\,, & \{ff'\}^3 = 0\,, \\
\{HH'\} = 0\,, & \{HH'\}^2 = m_2 D\,, & \\
\{fH\} = m_3 J\,, & \{fH\}^2 = 0\,, & \\
\{JJ'\} = 0\,, & \{JJ'\}^2 = m_4 H_1 D\,, & \{JJ'\}^3 = 0\,, \\
\{HJ\} = m_5 D_1 f\,, & \{HJ\}^2 = 0\,, & \\
\{fJ\} = m_6 H^2\,, & \{fJ\}^2 = 0\,, & \{fJ\}^3 = m_7 D\,,
\end{array}
$$

in which

$$
f = f' = \alpha\beta\gamma\,, \quad H = H' = \Sigma(\alpha\beta)^2\gamma^2\,, \quad J = J' = \Sigma(\alpha\beta)^2(\alpha\gamma)\,\beta\gamma^2\,,
$$

and $m_1, m_2, m_3, \ldots$ are numerical multipliers.

In this table and throughout this paper, the names of forms, as H and J, refer to functions of the roots; it is evident that the transvectants give numerical multiples of these functions.

10. *The connection between coefficient and root symbols.* It has been shown by Gordan‡ that the symbolic coefficient forms may be derived pri-

* See general invariant and covariant as given in Arts. 2 and 3.

† See Gordan's Vorlesungen, Vol. II, p. 172.

‡ Gordan's Vorlesungen, Vol. II, pp. 10, 34, 39.

marily from a consideration of root symbols. Thus, letting

$$\varphi = \rho_1\rho_2\rho_3 \ldots = a_x^m ,$$
$$\psi = \sigma_1\sigma_2\sigma_3 \ldots = b_x^n ,$$

in which $\rho_1, \rho_2, \rho_3, \ldots$ represent $\alpha, \beta, \gamma, \ldots$ and are made identical after transvection, and $\rho_1, \rho_2, \rho_3, \ldots = \sigma_1, \sigma_2, \sigma_3, \ldots$, we obtain by means of Art. 8

$$\{\varphi\psi\}^k = \Sigma(\rho_1\sigma_1) \ldots (\rho_k\sigma_k) \frac{\varphi\psi}{\rho_1 \ldots \rho_k \cdot \sigma_1 \ldots \sigma_k} .$$

Substituting a for every ρ_r and b for every σ_s,

$$\{\varphi\psi\}^k = m(m-1)\ldots(m-k+1)\, n(n-1)\ldots(n-k+1)(ab)^k a^{m-k} b^{n-k} ;$$

$$\therefore \quad (ab)^k a^{m-k} b^{n-k} = \frac{1}{m(m-1)\ldots(m-k+1) \,.\, n(n-1)\ldots(n-k+1)} \{\varphi\psi\}^k .$$

In this view, the coefficient symbol is a reduced form of the root symbol. Any transvectant operator applied to given covariant functions in their root forms, produces a certain other covariant or invariant expressed in the roots; and the same covariant or invariant in its coefficient symbol is produced by the application of this same transvectant operator upon the given covariants in their coefficient symbols. Symbolic covariants and invariants whether expressed in root or coefficient form are thus but developed transvectants. If $\{ff'\}^2$ be applied to $a^4 = b^4$, the form $(ab)^2a^2b^2$ is obtained; if applied to $\alpha\beta\gamma\delta\ [= a^4]$, the form $\Sigma(\alpha\beta)^2\gamma^2\delta^2$ is obtained; the first result being the coefficient symbol for H of the quartic, the second result the root symbol* of the same invariant.

11. *Transvectant relations among covariants and invariants.* Since the root and coefficient symbols of a covariant or invariant are different expressions for the same transvectant, all the transvectant relations which Clebsch and Gordan† have proved to exist among covariants and invariants in their symbolic coefficient forms are equally applicable to the symbolic root forms of covariants and invariants. The same relations can, of course, be proved independently for root forms, but with greater labor, since the root expression, equivalent to a single-termed coefficient symbol may involve many terms.

12. *Root symbols obtained by transvection.* As an example of the operation of transvection, and to illustrate the process of obtaining root symbols, we

* For the quantic $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$, this root symbol becomes $\Sigma(\alpha-\beta)^2 (x-\gamma)^2(x-\delta)^2$.

† See Clebsch Theorie der binären etc., Gordan's Vorlesungen.

give the following derivation of the root forms of the covariants H and J of the cubic $\phi = \varphi = \alpha\,{}_\iota\beta\gamma = (\alpha_1 x + \alpha_2 y)({}_\iota\beta_1 x + {}_\iota\beta_2 y)(\gamma_1 x + \gamma_2 y)$:—

$$\{\varphi\phi\}^2 = 2\left\{\frac{d^2\varphi}{dx^2}\frac{d^2\varphi}{dy^2} - \left[\frac{d^2\varphi}{dx\,dy}\right]^2\right\} \qquad \text{(See Art. 8.)} \qquad (1)$$

$$= 2(\alpha\,{}_\iota\beta)^2\left\{\frac{d^2\varphi}{d\alpha^2}\frac{d^2\varphi}{d\,{}_\iota\beta^2} - \left[\frac{d^2\varphi}{d\alpha\,d\,{}_\iota\beta}\right]^2\right\} + 2(\gamma\alpha)^2\left\{\frac{d^2\varphi}{d\gamma^2}\frac{d^2\varphi}{d\alpha^2} - \left[\frac{d^2\varphi}{d\gamma\,d\alpha}\right]^2\right\}$$

$$+ 2({}_\iota\beta\gamma)^2\left\{\frac{d^2\varphi}{d\,{}_\iota\beta^2}\frac{d^2\varphi}{d\gamma^2} - \left[\frac{d^2\varphi}{d\,{}_\iota\beta\,d\gamma}\right]^2\right\}$$

$$+ 4(\alpha\,{}_\iota\beta)(\gamma\alpha)\left\{\frac{d^2\varphi}{d\alpha\,d\gamma}\frac{d^2\varphi}{d\alpha\,d\,{}_\iota\beta} - \frac{d^2\varphi}{d\,{}_\iota\beta\,d\gamma}\frac{d^2\varphi}{d\alpha^2}\right\}$$

$$+ 4(\alpha\,{}_\iota\beta)({}_\iota\beta\gamma)\left\{\frac{d^2\varphi}{d\,{}_\iota\beta\,d\gamma}\frac{d^2\varphi}{d\alpha\,d\,{}_\iota\beta} - \frac{d^2\varphi}{d\alpha\,d\gamma}\frac{d^2\varphi}{d\,{}_\iota\beta^2}\right\}$$

$$+ 4(\gamma\alpha)({}_\iota\beta\gamma)\left\{\frac{d^2\varphi}{d\,{}_\iota\beta\,d\gamma}\frac{d^2\varphi}{d\alpha\,d\gamma} - \frac{d^2\varphi}{d\alpha\,d\,{}_\iota\beta}\frac{d^2\varphi}{d\gamma^2}\right\}$$

$$= -2\{(\alpha\,{}_\iota\beta)^2\gamma^2 + (\gamma\alpha)^2\,{}_\iota\beta^2 + ({}_\iota\beta\gamma)^2\alpha^2 - 2(\alpha\,{}_\iota\beta)(\gamma\alpha)\,{}_\iota\beta\gamma - 2(\alpha\,{}_\iota\beta)({}_\iota\beta\gamma)\,\alpha\gamma$$
$$- 2(\gamma\alpha)({}_\iota\beta\gamma)\,\alpha\,{}_\iota\beta\};$$

but by formula (3) Art. 5

$$2(\alpha\,{}_\iota\beta)(\gamma\alpha)\,{}_\iota\beta\gamma = \alpha^2({}_\iota\beta\gamma)^2 - {}_\iota\beta^2(\gamma\alpha)^2 - \gamma^2(\alpha\,{}_\iota\beta)^2,$$

$$\therefore \quad \{\varphi\phi\}^2 = -4\{(\alpha\,{}_\iota\beta)^2\gamma^2 + (\gamma\alpha)^2\,{}_\iota\beta^2 + ({}_\iota\beta\gamma)^2\alpha^2\}$$
$$= -4\Sigma(\alpha\,{}_\iota\beta)^2\gamma^2;$$

$\therefore\ \Sigma(\alpha\,{}_\iota\beta)^2\gamma^2$ is the root symbol for H, which has the coefficient symbol $(ab)^2ab$.

$$\{\varphi H\} = (\alpha\,{}_\iota\beta)\left\{\frac{d\varphi}{d\alpha}\frac{dH}{d\,{}_\iota\beta} - \frac{d\varphi}{d\,{}_\iota\beta}\frac{dH}{d\alpha}\right\} + (\gamma\alpha)\left\{\frac{d\varphi}{d\gamma}\frac{dH}{d\alpha} - \frac{d\varphi}{d\alpha}\frac{dH}{d\gamma}\right\} \qquad (2)$$

$$+ ({}_\iota\beta\gamma)\left\{\frac{d\varphi}{d\,{}_\iota\beta}\frac{dH}{d\gamma} - \frac{d\varphi}{d\gamma}\frac{dH}{d\,{}_\iota\beta}\right\}$$

$$= 2(\alpha\,{}_\iota\beta)\{(\gamma\alpha)^2\,{}_\iota\beta^2\gamma - ({}_\iota\beta\gamma)^2\alpha^2\gamma\} + 2(\gamma\alpha)\{({}_\iota\beta\gamma)^2\alpha^2\,{}_\iota\beta - (\alpha\,{}_\iota\beta)^2\,{}_\iota\beta\gamma^2\}$$
$$+ 2({}_\iota\beta\gamma)\{(\alpha\,{}_\iota\beta)^2\alpha\gamma^2 - (\gamma\alpha)^2\alpha\,{}_\iota\beta^2\}$$
$$= 2\Sigma(\alpha\,{}_\iota\beta)^2({}_\iota\beta\gamma)\alpha\gamma^2.$$

Practice in developing transvectants and in applying the general results of Art. 8 would make it evident upon inspection that $\{\varphi\phi\}^2$ is some numerical multiple of $\Sigma(\alpha\,{}_\iota\beta)^2\gamma^2$, and that $\{\varphi H\}$ is such a multiple of $\Sigma(\alpha\,{}_\iota\beta)^2({}_\iota\beta\gamma)\alpha\gamma^2$.

13. *Numerical relations between root and coefficient symbols.* In the results of a calculation of root forms which are given in Chap. III, no attempt

is made to express the numerical relations between the root functions and their corresponding coefficient functions, or the relations between root functions and their transvectants. To obtain such relations between a coefficient form and its root expression involves a consideration of the transvectant multiplier (Art. 10) and of the combinations that occur in the reductions to the irreducible form of the root function. And to establish relations between the root expression and the coefficient function itself (not its symbol) involves the determining of the numerical relation between the Clebsch symbol and the coefficient function; or the determining of relations between the coefficient function and the root expression by means of known coefficient and root relations. The following is a simple example of the numerical relation existing between the coefficient function (Salmon's), the coefficient symbol, and the root symbol :—

$$H\,(\text{Salmon}) = \tfrac{1}{2}\,H\,(\text{Clebsch}) = \frac{1}{2n^2(n-1)^2}\{ff\}^2$$

$$= \frac{1}{2n^2(n-1)^2}\{-2(n-1)\,\Sigma(\alpha_i\beta)^2\gamma^2\delta^2\ldots\}$$

$$= -\frac{1}{n^2(n-1)}\,\Sigma(\alpha_i\beta)^2\gamma^2\delta^2\ldots.$$

Chapter II.

A Comparison of Root and Coefficient Symbols.

14. *The covariant coefficient symbol.* The covariant and invariant coefficient symbols, called by Salmon the symbols of Aronhold* and Clebsch, are the expression of transvectants of the quantic a^n, which is $(a_1x + a_2y)^n$; and they have been shown to be the reduced forms of root symbols (See Art. 10). The coefficient symbol for a covariant is of the form†

$$(ab)^{e_{12}}(ac)^{e_{13}}(bc)^{e_{23}}\ldots(lm)^{e_{rs}}a^{e_1}b^{e_2}c^{e_3}\ldots,$$

where $e_r + e_{r.1} + e_{r,2} + e_{r,3} + \ldots + e_{r,r-1} + e_{r,r+1} + \ldots e_{r,n} = n$, n being the degree of the quantic to which the covariant belongs, the number of different letters denoting the order of the function, the number of linear factors the degree, and the number of determinant factors the weight. Thus J of the quartic is denoted by $(ab)^2(ac)\,ab^2c^3$, which is of the degree 6, order 3, and weight 3, and in its symbol each letter appears four times.

* Aronhold, Crelle's Journal, Vol. LXII, 281.

† See Clebsch, Gordan, Salmon.

Osgood "On the symbolic notation of Aronhold and Clebsch." American Journal, Vol. 14, p. 251.

15. *Order, weight and degree in root and coefficient symbols.* The degree and weight in the root symbols are determined as indicated in Art. 4; the degree and weight in the coefficient symbols are determined similarly. The number of different letters in the root symbol of a covariant equals the number of times any one letter appears in its coefficient symbol; while the number of times any one letter appears in the root symbol equals the number of different letters in the coefficient symbol. Evidently an operator whose effect is to substitute for each letter $a, b, \ldots$ of the coefficient symbol, one α, one $\beta, \ldots$ will change a given covariant or invariant coefficient symbol into a covariant or invariant root symbol of the same order, weight, and degree as the given coefficient form. If the given form be known to be an irreducible* form in the system of the quantic to which it belongs, and the only irreducible form of the given order and degree, the root form derived by means of such an operator will be either equal to the given form, or differ from it by a numerical factor.

16. *An operator which changes the coefficient symbol of an irreducible form into the root symbol of the same form.* Such an operator as that mentioned in the preceding article, is suggested by a comparison of the two forms of the quantic a^n,

$$(a_1x + a_2y)^n = (\alpha_1x + \alpha_2y)(\beta_1x + \beta_2y)(\gamma_1x + \gamma_2y)\ldots. \qquad (1)$$

Substituting $\dfrac{d}{da_1}$ for x and $\dfrac{d}{da_2}$ for y in the right hand member of the identity (1), there results the operator

$$\left[\alpha_1\frac{d}{da_1} + \alpha_2\frac{d}{da_2}\right]\left[\beta_1\frac{d}{da_1} + \beta_2\frac{d}{da_2}\right]\left[\gamma_1\frac{d}{da_1} + \gamma_2\frac{d}{da_2}\right]\ldots, \qquad (2)$$

which will be written

$$\left[\alpha\frac{d}{da}\right]\left[\beta\frac{d}{da}\right]\left[\gamma\frac{d}{da}\right]\ldots \qquad (3)$$

and denoted by the symbol $[D_a]$.

The operator (2) acting upon a^n or $(a_1x + a_2y)^n$ gives

$$n!\,(\alpha_1x + \alpha_2y)(\beta_1x + \beta_2y)(\gamma_1x + \gamma_2y)\ldots,$$

which we may write in the form

$$n!\,\alpha\beta\gamma\ldots.$$

In the case of two or more letters in the coefficient symbol, the action of the

* An *irreducible form* is one that cannot be expressed as a rational and integral function of other forms. Such a form has been also called a *fundamental form* by Salmon, and a *ground form* by Sylvester.

operator must be repeated for each letter.* The operator for any number of letters, a, b, c, . . . may be written in the form

$$[D_a \,.\, D_b \,.\, D_c \ldots] \,, \tag{4}$$

The operator (4) is to be interpreted as an ordinary differential operator, in which each operating factor $[D_a]$ acts either upon the original function or upon the result of the action of another operating factor or factors, while only one of these factors acts upon the original function. All combinations of the n roots that are possible within the limits of the given order, weight, and degree are admitted to the resulting root form. All of the combinations of the n roots which do not disappear, give terms which are of the same order, weight, and degree and which are, therefore, reducible to the same form. In any case this operator completely accomplishes the desired purpose of changing each coefficient letter into the n root letters, preserving the invariant form in its order, weight, and degree; and if in the form-system of the $n$ic there be but one irreducible form of the given order, weight, and degree, the result of the operation must be the root symbol of the given coefficient form.

A few examples are given:

(1) Let H† $= (ab)^2$,

$$\left[a \frac{d}{da} \right] H = 2\,(ab)\,(ab)\,, \quad [D_a]\, H = 2\,(ab)\,({}_{,}\beta b) \qquad A\,,$$

$$\left[a \frac{d}{db} \right] A = 2\,(ab)\,({}_{,}\beta a)\,, \quad [D_b]\, A = -\,2\,(a_{,}\beta)^2$$

$$\therefore\ [D_a D_b]\, H = -\,2\,(a_{,}\beta)^2$$

$\therefore$ $(a_{,}\beta)^2$ is the root symbol corresponding to $(ab)^2$.

(2) Let J‡ $= (ab)^2\,(ac)\,bc^2$

$$\tfrac{1}{2}\,[D_a]J = (ab)\,({}_{,}\beta b)\,(c\gamma)\,bc^2 + (ab)\,(\gamma b)\,(c_{,}\beta)\,bc^2 + (\gamma b)\,({}_{,}\beta b)\,(ca)\,bc^2 \qquad A_1\,,$$

$$\begin{aligned} \tfrac{1}{2}\,[D_b]A_1 = {} & (a_{,}\beta)\,({}_{,}\beta a)\,(c\gamma)\,\gamma c^2 + (a\gamma)\,({}_{,}\beta a)\,(c\gamma)\,{}_{,}\beta c^2 + (a_{,}\beta)\,({}_{,}\beta\gamma)\,(c\gamma)\,ac^2 \\ & + (a_{,}\beta)\,(\gamma a)\,(c_{,}\beta)\,\gamma c^2 + (a\gamma)\,(\gamma a)\,(c_{,}\beta)\,{}_{,}\beta c^2 + (a\gamma)\,(\gamma_{,}\beta)\,(c_{,}\beta)\,ac^2 \\ & + (\gamma a)\,({}_{,}\beta\gamma)\,(ca)\,{}_{,}\beta c^2 + (\gamma_{,}\beta)\,({}_{,}\beta a)\,(ca)\,\gamma c^2 + (\gamma_{,}\beta)\,({}_{,}\beta\gamma)\,(ca)\,ac^2 = A_2\,, \end{aligned}$$

$$[D_c]A_2 = -\,2\,.\,4\,\Sigma\,(a_{,}\beta)^2\,(a\gamma)\,\gamma^2{}_{,}\beta$$

* See Gordan's Vorlesungen, p. 41.

† Clebsch's H.

‡ Clebsch's J.

$\therefore\ \Sigma(\alpha\beta)^2(\alpha\gamma)\gamma^2\beta$ is the root symbol corresponding to the coefficient symbol $(ab)^2(ac)bc^2$.

17. *An operator which changes the root symbol of an irreducible form into the coefficient symbol of the same form.* The considerations which have given rise to an operator upon the coefficient forms, also give rise to an operator which changes root symbols into coefficient symbols. In the derivation of root symbols from coefficient symbols, the object is to introduce an a for one α, an a for one β, etc. for each root; this object is accomplished by the operator

$$\left[a_1\frac{d}{d\alpha_1}+a_2\frac{d}{d\alpha_2}\right]\left[a_1\frac{d}{d\beta_1}+a_2\frac{d}{d\beta_2}\right]\left[a_1\frac{d}{d\gamma_1}+a_2\frac{d}{d\gamma_2}\right]\ldots \tag{1}$$

repeated as many times as any root α appears in the root symbol, where a is replaced by a different letter $b, c, \ldots$ at each repetition. This operator (1) we write in the form

$$\left[a\frac{d}{d\alpha}\right]\left[a\frac{d}{d\beta}\right]\left[a\frac{d}{d\gamma}\right]\ldots, \tag{2}$$

and denote it by the symbol $[aD]$. A few applications of the operator are given:

(1) Let $H=(\alpha\beta)^2\gamma^2+(\alpha\gamma)^2\beta^2+(\beta\gamma)^2\alpha^2$,

$$[aD]H=\left[a\frac{d}{d\alpha}\right]\left[a\frac{d}{d\beta}\right]\left[a\frac{d}{d\gamma}\right]H$$

$$=4\left[(a\alpha)(\alpha\beta)\gamma a+(a\alpha)(\alpha\gamma)\beta a+(\beta a)(\alpha\gamma)aa\right]\equiv A_1,$$

$$[bD]A_1=-12(ab)^2ab$$

$\therefore\ (ab)^2ab$ is the coefficient symbol of the form whose root symbol is $\Sigma(\alpha\beta)^2\gamma^2$.

(2) Let $\qquad S=(\alpha\beta)^2(\gamma\delta)^2+(\alpha\gamma)^2(\beta\delta)^2+(\alpha\delta)^2(\beta\gamma)^2$,

then $\qquad [aD]S=12(a\alpha)(\alpha\beta)(\gamma a)(a\delta)\equiv A_1$,

$$[bD]A_1=12(ab)^4,$$

$\therefore\ (ab)^4$ is the coefficient symbol of $\Sigma(\alpha\beta)^2(\gamma\delta)^2$.*

This operator is of use only in the determination of the coefficient symbol of an irreducible form, which in its order and degree is unlike any other form in the form-system of its quantic, for the same reasons that operator (4), Art. 16, is of use only when applied to forms of this kind.

* Compare Gordan's Vorlesungen, p. 41.

18. *Derivation of root symbols from coefficient symbols.* In obtaining root symbols from coefficient symbols, a simplification of the process indicated in operator (4), Art. 16, is desirable. According to the principle of the formation of the operator, we can write as many root terms of given order, weight, and degree, as there are possible arrangements among the different roots within the imposed conditions. Each of these arrangements gives a term, the summation of which is an invariant function of the roots and of the given order, weight, and degree; not including the terms which disappear separately. If the given coefficient symbol be an irreducible form and the only irreducible form of the given order and degree among the forms belonging to its quantic, this coefficient form can differ from any one of the root summations only by a numerical factor; i. e. the different possible arrangements of the roots give different root symbols for the given invariant form. But all of these root symbols can be expressed as multiples of the most compact form—the standard form—of the given order, weight, and degree; for example, $\Sigma(\alpha\beta)(\alpha\gamma)\beta\gamma$ and $\Sigma(\alpha\beta)^2\gamma^2$ are forms derived from $(ab)^2ab$, and by the reduction formulæ (Art. 5) $\Sigma(\alpha\beta)(\alpha\gamma)\beta\gamma$ can be reduced to $\Sigma(\alpha\beta)^2\gamma^2$ which is the standard form.

The operation of changing from a coefficient symbol to a root symbol is (essentially) introducing an α for each letter, a β for each letter, and so on for the n roots $\alpha, \beta, \gamma, \ldots.$ If this be done, no matter in what order, it will give a form which will either be identically zero or be equal to some multiple of the standard root form of the given invariant or covariant. The forms which vanish can be avoided easily in the process of change. Let us consider $(ab)^4(ac)bc^4$, one of the irreducible forms of the form-system of the quantic. Changing one a into α, one a into β, etc., there results

$$(\alpha b)(\beta b)(\gamma b)(\delta b)(\varepsilon c)bc^4\,; \tag{1}$$

changing the b's similarly and avoiding terms $(\alpha\alpha)$, we get

$$(\alpha\beta)^2(\gamma\delta)^2(\varepsilon c)\varepsilon c^4\,; \tag{2}$$

and changing the c's similarly,

$$(\alpha\beta)^2(\gamma\delta)^2(\varepsilon\alpha)\beta\gamma\delta\varepsilon^2\,, \tag{3}$$

which is a term of a covariant, and of the same order, weight, and degree as the given form. Therefore $\Sigma(\alpha\beta)^2(\gamma\delta)^2(\varepsilon\alpha)\beta\gamma\delta\varepsilon^2$ is some multiple of $(ab)^4(ac)bc^4$; and, moreover, this summation cannot be reduced to a more compact form. The operation here performed is evidently an abbreviation of the action of the complete coefficient operator of Art. 16. The use of those combinations which give most compactness and symmetry to the resulting form, decreases the

amount of reduction necessary to bring the resulting root symbol to the standard form. In the example given, (2) might have been written

$$(a\delta)(\beta a)(\gamma\beta)(\delta\gamma)(\varepsilon c)\,\varepsilon c^4\,, \tag{4}$$

from which arises

$$(a\delta)(\beta a)(\gamma\beta)(\delta\gamma)(\varepsilon a)\,\beta\gamma\delta\varepsilon^2\,. \tag{5}$$

and therefore $\Sigma(a\delta)(\beta a)(\gamma\beta)(\delta\gamma)(\varepsilon a)\beta\gamma\delta\varepsilon^2$ is some numerical multiple of $(ab)^4(ac)bc^4$. But $(a\delta)(\beta a)(\gamma\beta)(\delta\gamma)$ can be replaced by $(a\beta)^2(\gamma\delta)^2$ or $(a\gamma)^2(\beta\delta)^2$ or $(a\delta)^2(\beta\gamma)^2$ (see Art. 5), and therefore $(ab)^4(ac)bc^4$ is a multiple of $\Sigma(a\beta)^2(\gamma\delta)^2(\varepsilon a)\beta\gamma\delta\varepsilon^2$.

As another example of this method of obtaining root symbols, consider the form $(ab)^4(cd)^4(ac)(bd)$, which is what Salmon calls the J invariant of the quintic. Operating upon this form, we obtain

$$(ab)(\beta b)(\gamma b)(\delta b)(cd)^4(bd)(c\varepsilon) \tag{6}$$

$$(a\beta)^2(\gamma\delta)^2(cd)^4(\varepsilon d)(c\varepsilon) \tag{7}$$

$$(a\beta)^2(\gamma\delta)^2(ad)(\beta d)(\gamma d)(\varepsilon d)^2(\delta\varepsilon) \tag{8}$$

$$(a\beta)^4(\gamma\delta)^2(\gamma\varepsilon)^2(\varepsilon\delta)^2 \tag{9}$$

$$\therefore\ \Sigma(a\beta)^4(\gamma\delta)^2(\gamma\varepsilon)^2(\varepsilon\delta)^{2*} = m\,(ab)^4(cd)^4(bd)(ca) \tag{10}$$

where m is any numerical multiplier.

This method of deriving root symbols cannot be of use in determining the root symbol of a form which is not the only irreducible form of its order and weight in the form system of its quantic. In such a case, as in all cases, the root symbol can be obtained from other root symbols by transvection (see Art. 12). In most cases the transvection method is simpler than that of the above examples.

19. *Cayley symbols.*† The Cayley symbol indicates an actual operation to be performed upon any quantic or quantics, while the Aronhold and Clebsch symbol expresses the symbolic result of the operation indicated by the corresponding Cayley symbol. Thus $\overline{12}^2\,\overline{13}$ acting upon a quartic is the Cayley symbol for $(ab)^2(ac)\,ab^2c^3$; the Cayley operator being identical in its sym-

* Salmon gives this expression for the J invariant and remarks that the function $\Sigma(a-\beta)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\beta-\gamma)^2(\varepsilon-a)^2$ has been found to have the same value. According to the principles by which the above results are obtained, the form (10), and the function just given, and also the function $\Sigma(a-\beta)(\gamma-\delta)(a-\gamma)(\beta-\delta)(\delta-\varepsilon)^2(\beta-\gamma)^2(\varepsilon-a)^2$ are different expressions for the same function J (see Salmon, pp. 246, 259).

† Cayley, Collected Papers; *Crelle*, Vol. XXX. Salmon, M. H. A., p. 137.

bolic form with the determinant part of the Aronhold and Clebsch symbol. If a Cayley symbol for any covariant or invariant act upon any quantic in its coefficient form, the coefficient expression of the covariant or invariant is obtained; if the Cayley symbol act upon the quantic expressed as a product of linear factors the result will be the covariant or invariant expressed in terms of the roots of the quantic. E. g.

$$\overline{12}^2 = \begin{vmatrix} \dfrac{d}{dx_1} & \dfrac{d}{dy_1} \\ \dfrac{d}{dx_2} & \dfrac{d}{dy_2} \end{vmatrix}^2 = 2\left\{ \frac{d^2}{dx_1^2}\frac{d^2}{dy_1^2} - \left[\frac{d^2}{dx_1 dy_1}\right]^2 \right\}, \tag{1}$$

the operand being the product of the quantics $f(x_1, y_1)$, $f(x_2, y_2)$, which are to be made identical after the differentiations have been performed. Let

$$f(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3 = a(x - \alpha y)(x - \beta y)(x - \gamma y).$$

The operator (1) acting upon $ax^3 + 3bx^2y + \ldots$ in the manner stated produces

$$(ac - b^2)x^2 + (ad - bc)xy + (bd - c^2)y^2,$$

and acting upon $a(x - \alpha y)(x - \beta y)(x - \gamma y)$ produces

$$a^2\,\Sigma(\alpha - \beta)^2(x - \gamma y)^2.$$

20. *Systems of quantics.* If φ be a covariant of any quantic f, and ψ a covariant of a different quantic f', then $\{\varphi\psi\}^k$ is a covariant of the system of two quantics f and f'.

E. g. Let $\varphi = (ab)^2\,ab$, and $\psi = (a'b')^2(a'c')\,a'b'^2c'^3$; then $\{\varphi\psi\}^2$ is $(ab)^2(a'b')^2(a'c')(aa')(bb')\,b'c'^3$, which is a covariant of the system of cubic and quartic. Similarly, if $\varphi = \Sigma(\alpha\beta)^2\gamma^2$, and $\psi = \Sigma(\alpha'\beta')^2(\alpha'\gamma')\,\beta'\gamma'^2\delta'^3$, then $\{\varphi\psi\}^2$ produces $\Sigma(\alpha\beta)^2(\alpha'\beta')^2(\alpha'\gamma')(\gamma\gamma')^2\,\beta'\delta'^3$.

The typical covariant of the system f and f', when f is of the degree n and f' of the degree m, is (compare Art. 3)

$$\Sigma(\alpha\beta)^{e_{12}}\ldots(\mu\nu)^{e_{n-1,n}}(\alpha'\beta')^{e''_{12}}\ldots(\rho'\sigma')^{e''_{m-1,m}}(\alpha\alpha')^{e'_{11}}(\alpha\beta')^{e'_{12}}\ldots(\nu\sigma')^{e'_{nm}}\,\alpha^{e_1}\ldots\nu^{e_n}\,\alpha'^{e'_1}\ldots\sigma'^{e'_m},$$

wherein

$$\begin{array}{l}
e_1 + e_{12} + e_{13} + \ldots e_{1n} \qquad + e'_{11} + e'_{12} + \ldots e'_{1m} = \omega, \\
e_2 + e_{21} + e_{23} + \ldots e_{2n} \qquad + e'_{21} + e'_{22} + \ldots e'_{2m} = \omega, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
e_n + e_{n1} + e_{n2} + \ldots e_{n,n-1} + e'_{n1} + e'_{n2} + \ldots e'_{nm} = \omega, \\
e'_1 + e''_{12} + e''_{13} + \ldots e''_{1m} \quad + e'_{11} + e'_{12} + \ldots e'_{1n} = \omega', \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
e'_m + e''_{m1} + e''_{m2} + \ldots e''_{m,m-1} + e'_{m1} + e'_{m2} + \ldots e'_{mn} = \omega'.
\end{array}$$

Covariant and invariant relations among any number of quantics can be similarly expressed. The operators of Arts. 16, 17 may be applied to these forms as to the forms of a single quantic, the action of the operator being repeated for each quantic of the system; and the coefficient symbol of a covariant or invariant of a system of quantics may be changed into a root symbol by the methods of Art. 18, if the given form be an irreducible form unique in its order and degree. If a covariant or invariant of a system of two quantics be represented by a Cayley symbol, the particular quantic to which each figure 1, 2, . . . refers, must be indicated; unless the covariant or invariant of the two quantics is of the first order in regard to each quantic, i. e. unless $\omega = \omega' = 1$. Thus $(ab)^2\,(aa')\,a'^2b$ is completely represented by $\overline{12}^2\,\overline{13}\,.\,U_1^{(3)}\,U_2^{(3)}\,U_3'^{(3)}$, in which the indices of U denote the degree of the quantic, and $U_1^{(3)}$, $U_2^{(3)}$ are to be made identical; but if the quantics operated upon be not so defined, the Cayley symbol of this covariant can not be distinguished from that of $(a'b')^2\,(aa')\,a^2b'$ (see Table IX, Nos. 4, 6).

Similarly, the two forms $(ab)^2\,(aa')\,(bb')$ and $(ab)^2\,(ac)\,(ba')\,c^2$ are $\overline{12}^2\,\overline{13}\,\overline{24}\,.\,U_1^{(3)}\,U_2^{(3)}\,U_3'^{(1)}\,U_4'^{(1)}$ and $\overline{12}^2\,\overline{13}\,\overline{24}\,.\,U_1^{(3)}\,U_2^{(3)}\,U_3^{(3)}\,U_4'^{(1)}$. The form $(a'a)\,a^2$ may be represented by $\overline{12}$ acting upon the linear and cubic, in which it is indifferent to which quantic either figure refers; and $(a'a)^2a$ may be represented by $\overline{12}^2$ acting upon a cubic and quadratic.

Similarly, in the Cayley symbol of a form of a system of k quantics, the quantics to which each figure refers, must be definitely indicated unless the form is of the first order in regard to each of the k quantics.

CHAPTER III.

TABLES OF COVARIANTS AND INVARIANTS.

21. The Tables at the end of this memoir include the form-system of the lower quantics through the sextic, and the form-systems of pairs of the first five quantics (including the linear quantic); and afford a complete list of the root expressions of the irreducible covariants and invariants of these quantics and systems of quantics. The arrangement of the forms is such as to facilitate a comparison between root symbols and coefficient symbols, and a comparison of the corresponding forms of different form-systems. The relations among the different forms of any system are indicated by the transvectants. The root symbols have been obtained from the root symbols of the quantics through transvection. (See Arts. 12, 8.) The Cayley symbols have been obtained

directly from the Aronhold and Clebsch symbols; while the Aronhold and Clebsch symbols are the symbolic coefficient expressions of the transvectants. Many of the transvectants are those given by Clebsch; and a number of others have been formed from known relations among the forms involved in them. The transvectants of Tables V and VI have been obtained from considerations of the orders and degrees given by Sylvester in his Tables* of ground forms. The transvectants of Table X were obtained by an inspection of the forms given by Gundelfinger in his *Programme*† of 1869.

The names given by Salmon are used in the Tables as far as possible; when the Salmon names have been lacking, those of Clebsch are used. In the few cases in which both are given, the first is Salmon's name, and the second that of Clebsch. Forms named neither by Salmon nor Clebsch have been named according to some characteristic, or by analogy; or if a suggestion has been wanting, the names C_1, C_2, . . . have been given to the unnamed covariants and I_1, I_2, . . .‡ to the unnamed invariants.

In the Tables of systems of simultaneous quantics, those covariants and invariants which belong to each quantic separately are not given, but may be found in the Tables of the single quantics. The Cayley symbols of forms belonging to a system of quantics are given only for those forms which are of the first order in regard to each quantic. The Cayley symbols of forms of higher orders involve a designation of the quantics to be operated upon (see Art. 20); it has been thought needless to give these symbols, since the forms they represent are completely defined by the simpler Aronhold and Clebsch symbols, and since all Cayley symbols can be obtained directly from the Aronhold and Clebsch symbols.

22. *Root differences.* It is desired to emphasize the fact that the symbolic root expression can be written as a symmetric function of root differences (see Art. 2); for this reason the root symbols and root differences are written side by side in Table I. When the general quantic

$$(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)\ldots \tag{1}$$

is written in the form

$$a_0(x-\alpha)(x-\beta)(x-\gamma)\ldots, \tag{2}$$

* Sylvester, *American Journal of Math.*, Vol. II.

† Gundelfinger, "*Zur Theorie des simultanen Systems einer cubischen und einer biquadratischen binären Forms*," Stuttgart, 1869.

Sylvester has shown that three of the forms given by Gundelfinger are reducible forms; these are Gundelfinger's forms $(a\pi)(a\sigma)a$, $(Tw)^2(Tw')^2T^2$, and $(aT)^3(TQ)^2QT$ (see Sylvester, *Comptes Rendus* 87).

‡ This is somewhat in accordance with the nomenclature of Faà de Bruno.

y of (1) becomes unity, a_2 of (1) is replaced by $-aa_1$, and $a_1\beta_1\gamma_1 \ldots$ by a_0; and the general covariant (see Art. 2)

$$\Sigma(a\beta)^{e_{12}}(\beta\gamma)^{e_{23}}\ldots(\mu\nu)^{e_{n-1,n}}\,.\,a^{e_1}\beta^{e_2}\ldots\nu^{e_n}, \qquad (3)$$

which is

$$\Sigma(a_1\beta_2 - a_2\beta_1)^{e_{12}}(\beta_1\gamma_2 - \beta_2\gamma_1)^{e_{23}}\ldots(\mu_1\nu_2 - \mu_2\nu_1)^{e_{n-1,n}}$$

$$.\,(a_1x + a_2y)^{e_1}(\beta_1x + \beta_2y)^{e_2}\ldots(\nu_1x + \nu_2y)^{e_n}, \qquad (4)$$

becomes

$$a_0^{\omega}\,\Sigma(a-\beta)^{e_{12}}(\beta-\gamma)^{e_{23}}\ldots(\mu-\nu)^{e_{n-1,n}}(x-a)^{e_1}(x-\beta)^{e_2}\ldots(x-\nu)^{e_n}. \qquad (5)$$

Thus H of the quintic has the root symbol $\Sigma(a\beta)^2\gamma^2\delta^2\varepsilon^2$, which expressed as a covariant of the quintic of the form (2), is

$$a_0^2\,\Sigma(a-\beta)^2(x-\gamma)^2(x-\delta)^2(x-\varepsilon)^2,$$

and K of the same quintic has the symbol

$$\Sigma(a\beta)^6(\gamma\delta)^6(a\varepsilon)^2(\beta\varepsilon)^2(\gamma\varepsilon)^2(\delta\varepsilon)\ ,$$

and may be written

$$a_0^8\,\Sigma(a-\beta)^6(\gamma-\delta)^6(a-\varepsilon)^2(\beta-\varepsilon)^2(\gamma-\varepsilon)^2(\delta-\varepsilon)^2.$$

The quantic $(x-a)(x-\beta)(x-\gamma)\ldots$ is a particular form of the quantic (2) in which a_0 equals unity; and for which the covariant (3) becomes

$$\Sigma(a-\beta)^{e_{12}}(\beta-\gamma)^{e_{23}}\ldots(\mu-\nu)^{e_{n-1,n}}\,.\,(x-a)^{e_1}(x-\beta)^{e_2}\ldots(x-\nu)^{e_n},$$

where each factor $(a\beta)$ in the covariant symbol is replaced by $(a-\beta)$ and each linear factor a, which is $a_1x + a_2y$, is replaced by $(x-a)$. This latter form of the quantic, the form $(x-a)(x-\beta)(x-\gamma)\ldots$, is adopted in the tables, where the root differences are given. On the other hand the root symbols are the symbolic expressions of the root forms of the quantic in its more general form.

Chapter IV.

Particular Classes of Forms and Operators.

23. *Emanants.* Emanants of a binary quantic f, have been defined* as the functions $\left[x'\dfrac{d}{dx} + y'\dfrac{d}{dy}\right]^k f$, these functions being the coefficients of λ in

$$f(x+\lambda x', y+\lambda y') = f(x,y) + \lambda\Delta f(x,y) + \frac{\lambda^2}{1\,.\,2}\Delta^2 f(x,y) + \ldots, \qquad (1)$$

wherein $\Delta = x'\dfrac{d}{dx} + y'\dfrac{d}{dy}$.

* Salmon's M. H. Algebra, p. 115.

Emanants satisfy all the conditions of covariants (or invariants) of a system of $n$ic and 1ic (Art. 20); but are not properly covariants of the $n$ic, if covariants of the $n$ic be defined as in Art. 3.

In the operator Δ let $\alpha_1' = y'$ and $-\alpha_2' = x'$, then the emanant operator $\left[\alpha_1' \frac{d}{dy} + \alpha_2' \frac{d}{dx}\right]^k$ is in a form that corresponds with the notation of the preceding articles. The emanants of the quantic

$$(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)\dots$$

are as follows:

1st emanant $= \Delta\,[(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)\dots]$ (2)

$= \Sigma\,(\alpha_1'\alpha_2 - \alpha_2'\alpha_1)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)$ (3)

$= \Sigma\,(\alpha'\alpha)\,\beta\gamma\dots;$ (4)

2d emanant $= \Delta^2\,[\alpha\beta\gamma\dots]$ (5)

$= \Sigma\,(\alpha'\alpha)(\alpha'\beta)\,\gamma\delta\dots;$ (6)

3d emanant $= \Delta^3\,[\alpha\beta\gamma\dots]$ (7)

$= \Sigma\,(\alpha'\alpha)(\alpha'\beta)(\alpha'\gamma)\,\delta\dots;$ (8)

and similarly the higher emanants may be found; the kth emanant of the $n$ic being made up of terms in each of which are k determinant factors of the form $(\alpha'\beta)$. The nth emanant of an $n$ic is an invariant, and a quantic in α_1'/α_2' which has the same roots as the original $n$ic (see Tables V and VI).

The emanant of any covariant is similarly obtained and is a covariant of the system of the given covariant and linear. Thus, for the 3d emanant of H of the $n$ic

$$\Delta^3 H = \Delta^3\,[\Sigma\,(\alpha\beta)^2\gamma^2\delta^2\varepsilon^2\dots]$$

$$= \Sigma\,(\alpha\beta)^2(\alpha'\gamma)^2(\alpha'\delta)\,\delta\varepsilon^2\dots;$$

and for the 2nd emanant of J of the $n$ic,

$$\Delta^2 J = \Delta^2\,[\Sigma\,(\alpha\beta)^2(\alpha\gamma)\,\beta\gamma^2\delta^3\varepsilon^3\dots]$$

$$= \Sigma\,(\alpha\beta)^2(\alpha\gamma)(\alpha'\gamma)^2\,\beta\delta^3\varepsilon^3\dots.$$ (See Tables V and VI.)

An inspection of the Tables given by Sylvester* shows that the emanants of the quantics there considered and the emanants of the irreducible covariants of those quantics are themselves irreducible forms. Emanants are polars, and are treated as such in Art. 51.

* Sylvester, American Journal of Mathematics, Vol. II.

24. *Evectants.* Evectants of a binary $n$ic are the functions obtained by the action of the operator

$$\left[\xi^n \frac{d}{da_0} + \xi^{n-1}\eta \frac{d}{da_1} + \xi^{n-2}\eta^2 \frac{d}{da_2} + \ldots\right] \tag{1}$$

upon any invariant of the $n$ic, where the $n$ic is of the general form

$$\bar{a}_0 x^n + n\bar{a}_1 x^{n-1}y + \frac{n(n-1)}{1\,.\,2}\bar{a}_2 x^{n-2}y^2 + \ldots,$$

and where ξ and η denote variables transformed by the inverse* substitution. The kth evectant is the result of k repetitions of the above operator. It is known that in any binary quantic y and $-x$ are cogredient* to ξ and η. Substituting y and $-x$ for ξ and η in (1) we obtain the evectant operator

$$\left[y^n \frac{d}{da_0} - y^{n-1}x \frac{d}{da_1} + y^{n-2}x^2 \frac{d}{da_2} - y^{n-3}x^3 \frac{d}{da_3} + \ldots\right]. \tag{2}$$

In the Clebsch symbolism (Art. 14)

$$(a_1 x + a_2 y)^n = \bar{a}_0 x^n + n\bar{a}_1 x^{n-1}y + \ldots + \frac{n!}{r!(n-r)!}\bar{a}_r x^{n-r}y^r + \ldots, \tag{3}$$

where

$$\bar{a}_r = a_1^{\,n-r}a_2^{\,r}, \tag{4}$$

and

$$\frac{d}{da_r} = \frac{1}{(n-r)!\,r!}\left[\frac{d}{da_1}\right]^{n-r}\left[\frac{d}{da_2}\right]^r. \tag{5}$$

Substituting from (5) in (2), we obtain the expression

$$\frac{1}{n!}\Bigg[\left[\frac{d}{da_1}\right]^n y^n - n\left[\frac{d}{da_1}\right]^{n-1}\left[\frac{d}{da_2}\right]xy^{n-1} + \ldots$$
$$+ (-1)^r \frac{n!}{(n-r)!\,r!}\left[\frac{d}{da_1}\right]^{n-r}\left[\frac{d}{da_2}\right]^r x^r y^{n-r} + \ldots\Bigg], \tag{6}$$

which is

$$\frac{1}{n!}\left[y\frac{d}{da_1} - x\frac{d}{da_2}\right]^n, \tag{7}$$

an operator which if acting upon the coefficient symbol of any binary invariant will produce the evectants of that invariant.† Omitting the numerical factor, (7) may be written in the form

$$\left[y\frac{d}{da_1} - x\frac{d}{da_2}\right]\left[y\frac{d}{d\beta_1} - x\frac{d}{d\beta_2}\right]\left[y\frac{d}{d\gamma_1} - x\frac{d}{d\gamma_2}\right]\ldots, \tag{8}$$

*Salmon's M. H. Algebra, pp. 118, 128.

†Operator (1) and the corresponding symbolic operator (7) applied to covariants produce other covariants. See Salmon, p. 122.

in which

$$\left.\begin{aligned}
&\frac{d}{d\alpha_1}\frac{d}{d{}_1\beta_1}\frac{d}{d\gamma_1}\cdots = \left[\frac{d}{d\alpha_1}\right]^n,\\
&\Sigma\left[\frac{d}{d\alpha_2}\frac{d}{d{}_1\beta_1}\frac{d}{d\gamma_1}\cdots\right] = -\,n\left[\frac{d}{d\alpha_1}\right]^{n-1}\left[\frac{d}{d\alpha_2}\right],\\
&\Sigma\left[\frac{d}{d\alpha_2}\frac{d}{d{}_1\beta_2}\frac{d}{d\gamma_1}\cdots\right] = \frac{n\,(n-1)}{2}\left[\frac{d}{d\alpha_1}\right]^{n-2}\left[\frac{d}{d\alpha_2}\right]^2.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\tag{9}$$

The operator (8) acting upon the root symbol of any invariant, produces the root symbols of the evectants of that invariant.

Let $\left[y\dfrac{d}{da_1} - x\dfrac{d}{da_2}\right]^n$, which is the operating factor in (7), act upon T of the quartic, whose coefficient symbol is $(ab)^2\,(ac)^2\,(bc)^2$; and let $\left[\dfrac{d}{da}\right]^n$ represent $\left[y\dfrac{d}{da_1} - x\dfrac{d}{da_2}\right]^n$; then*

$$\frac{1}{2}\left[\frac{d}{da}\right] T = (ab)\,(ac)^2\,(bc)^2\,(b_1x + b_2y) + \ldots = T_1\,,\tag{10}$$

$$\left[\frac{d}{da}\right] T_1 = (ac)^2\,(bc)^2\,(b_1x + b_2y)^2 + \ldots = T_2\,,\tag{11}$$

$$\frac{1}{6}\left[\frac{d}{da}\right] T_2 = (ac)\,(bc)^2\,(b_1x + b_2y)^2\,(c_1x + c_2y) + \ldots = T_3\,,\tag{12}$$

$$\frac{1}{2}\left[\frac{d}{da}\right] T_3 = (bc)^2\,(b_1x + b_2y)^2\,(c_1x + c_2y)^2 = (bc)^2b^2c^2 = H \text{ of quartic}.\tag{13}$$

Let the operator (8) act upon $\Sigma\,(\alpha\,{}_1\beta)^2\,(\gamma\delta)^2\,(\alpha\gamma)\,({}_1\beta\delta)$, the root symbol of T of the quartic; and let

$$\left[\frac{d}{d\alpha}\right] = \left[y\frac{d}{d\alpha_1} - x\frac{d}{d\alpha_2}\right],\quad \left[\frac{d}{d{}_1\beta}\right] = \left[y\frac{d}{d{}_1\beta_1} - x\frac{d}{d{}_1\beta_2}\right],\quad \ldots;$$

then

$$\frac{1}{2}\left[\frac{d}{d\alpha}\right] T = \Sigma\,(\alpha\,{}_1\beta)\,(\gamma\delta)^2\,(\alpha\gamma)\,({}_1\beta\delta)\,({}_1\beta_1x + {}_1\beta_2y) + \ldots = T_1\,,\tag{14}$$

$$\left[\frac{d}{d{}_1\beta}\right] T_1 = \Sigma\,(\gamma\delta)^2\,(\alpha\gamma)\,({}_1\beta\delta)\,(\alpha_1x + \alpha_2y)\,({}_1\beta_1x + {}_1\beta_2y) + \ldots = T_2\,,\tag{15}$$

$$\left[\frac{d}{d\gamma}\right] T_2 = \Sigma\,(\gamma\delta)^2\,({}_1\beta\delta)\,(\alpha_1x + \alpha_2y)^2\,({}_1\beta_1x + {}_1\beta_2y) + \ldots = T_3\,,\tag{16}$$

$$\left[\frac{d}{d\delta}\right] T_3 = \Sigma\,(\gamma\delta)^2(\alpha_1x + \alpha_2y)^2({}_1\beta_1x + {}_1\beta_2y)^2 + \ldots = M\Sigma\,(\gamma\delta)^2\alpha^2{}_1\beta^2 = H.\tag{17}$$

* Compare with Gordan's Vorlesungen, Vol. II, p. 127.

The operation upon the root forms is simplified by allowing the operating factors to act upon but one term of a summation. The numerical multipliers may be disregarded when it is only the symbol itself that is desired (see Art. 13).

If the general quantic be of the form

$$a_0(x-\alpha)(x-\beta)(x-\gamma)\ldots, \tag{18}$$

$\alpha_1\beta_1\gamma_1\ldots = a_0$, $\alpha_2 = -\alpha\alpha_1$, $\beta_2 = -\beta\beta_1$, $\ldots$, and y equals unity; when α_1, β_1, γ_1, $\ldots$ may be regarded as arbitrary constants whose product is a_0. Operator (8) adapted to the forms of quantic (18), becomes

$$\frac{1}{\alpha_1\beta_1\gamma_1\ldots}\left[x\frac{d}{d\alpha}\right]\left[x\frac{d}{d\beta}\right]\left[x\frac{d}{d\gamma}\right]\ldots,$$

which is

$$\frac{1}{a_0}\left[x\frac{d}{d\alpha}\right]\left[x\frac{d}{d\beta}\right]\left[x\frac{d}{d\gamma}\right]\ldots. \tag{19}$$

If (19) act upon any invariant of (18), expressed as a function of root differences, the evectants of that invariant are obtained; thus (19) acting upon $a_0^3\,\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)$, which is T of the quartic (18), gives $a_0^2\,\Sigma(\gamma-\delta)^2(x-\alpha)^2(x-\beta)^2$. In the quantic $(x-\alpha)(x-\beta)(x-\gamma)\ldots$, a_0 is unity; and the operator corresponding to (19) is $\left[x\frac{d}{d\alpha}\right]\left[x\frac{d}{d\beta}\right]\left[x\frac{d}{d\gamma}\right]$.

25. *Evectants of the discriminant.* We give the following proposition* and the demonstration of particular cases, to furnish examples of evectants and to make an interesting use of root expressions.

When the discriminant of a binary quantic vanishes, the quantic has a pair of equal roots, and the first evectant of the discriminant is of the form $(a'_1x + a'_2y)^n$, when $\frac{-a'_2}{a'_1}$ is one of the equal roots of the quantic; and if the quantic has k pairs of equal roots the kth evectant of the discriminant becomes of the form

$$(a'_1x + a'_2y)^n\,(a''_1x + a''_2y)^n\ldots(a_1^{(k)}x + a_2^{(k)}y)^n.$$

Thus the 1st evectant of D of the cubic, which is $(\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^2$, is $\Sigma(\alpha\beta)^2(\alpha\gamma)\beta\gamma^2$, and is J of the cubic; and each term of this evectant contains all but one of the root differences. If $D=0$, some one determinant, as $(\beta\gamma)$, vanishes and all but one term of the evectant disappear. The evectant equated to zero becomes

$$(\alpha\beta)^3\beta^3 = 0 \text{ or } \beta^3 = 0. \tag{1}$$

* This proposition is proved by Salmon for a ternary quantic. See M. H. Algebra, p. 123.

The discriminant of the quintic is

$$(\alpha\,{}_{i}\beta)^2\,(\alpha\gamma)^2\,(\alpha\delta)^2\,(\alpha\varepsilon)^2\,({}_{i}\beta\gamma)^2\,({}_{i}\beta\delta)^2\,({}_{i}\beta\varepsilon)^2\,(\gamma\delta)^2\,(\gamma\varepsilon)^2\,(\delta\varepsilon)^2\,; \tag{2}$$

and

$$\Sigma\,(\alpha\,{}_{i}\beta)^2\,(\alpha\gamma)^2\,(\alpha\delta)^2\,({}_{i}\beta\gamma)^2\,({}_{i}\beta\delta)^2\,(\gamma\delta)\,(\alpha\varepsilon)\,({}_{i}\beta\varepsilon)\,(\gamma\varepsilon)^2\,\delta^2\varepsilon^3 \tag{3}$$

is its first evectant ; and

$$\Sigma\,(\alpha\,{}_{i}\beta)^2\,(\alpha\gamma)\,(\alpha\delta)^2\,({}_{i}\beta\delta)\,(\gamma\delta)\,(\alpha\varepsilon)\,({}_{i}\beta\varepsilon)\,(\gamma\varepsilon)\,{}_{i}\beta^2\gamma^3\delta^2\varepsilon^3 \tag{4}$$

the second evectant. Each term of (3) contains all but one of the different determinants of (2). If (2) equal zero, some one determinant of (2), as $(\varepsilon\delta)$, vanishes, and (3) equated to zero becomes

$$(\alpha\,{}_{i}\beta)^2\,(\alpha\gamma)^2\,(\alpha\delta)^3\,({}_{i}\beta\gamma)^2\,({}_{i}\beta\delta)^3\,(\gamma\delta)^3\,\delta^5 = 0 \text{ or } \delta^5 = 0\,. \tag{5}$$

Each term of (4) contains all but two of the different determinants of (2); and if (2) equal zero and the quantic has two pairs of equal roots, then two of the determinants, as $(\varepsilon\delta)$ and $({}_{i}\beta\gamma)$, vanish, and (4) equated to zero becomes

$$(\alpha\,{}_{i}\beta)^3\,(\alpha\delta)^3\,({}_{i}\beta\delta)^4\,{}_{i}\beta^5\delta^5 = 0\,, \text{ or } {}_{i}\beta^5\delta^5 = 0\,, \tag{6}$$

which latter form may be written

$$({}_{i}\beta_1 x + {}_{i}\beta_2 y)^5\,(\delta_1 x + \delta_2 y)^5 = 0\,; \tag{7}$$

and similarly for the kth evectant of any $n$ic.

26. *Differential equations satisfied by covariants.* Covariants (including invariants, which are covariants of degree 0) have sometimes been defined as functions of the coefficients and variables of a quantic which satisfy the two differential equations*

$$a_0\frac{d\varphi}{da_1} + 2a_1\frac{d\varphi}{da_2} + 3a_2\frac{d\varphi}{da_3} + \ldots na_{n-1}\frac{d\varphi}{da_n} - y\frac{d\varphi}{dx} = 0\,, \tag{1}$$

$$a_n\frac{d\varphi}{da_{n-1}} + 2a_{n-1}\frac{d\varphi}{da_{n-2}} + 3a_{n-2}\frac{d\varphi}{da_{n-3}} + \ldots na_1\frac{d\varphi}{da_0} - x\frac{d\varphi}{dy} = 0\,. \tag{2}$$

Functions φ that satisfy equations (1) and (2), are functions which are unchanged by either of the transformations

$$x = \bar{x} + \lambda\bar{y}\,, \qquad y = 0.\bar{x} + \bar{y}\,; \tag{3}$$

$$x = \bar{x} + 0.\bar{y}\,, \qquad y = \lambda\bar{x} + \bar{y}\,; \tag{4}$$

in both of which the modulus is unity.

* Salmon's M. H. Algebra, p. 63. Gordan's Vorlesungen, p. 119, Vol. II. Burnside and Panton's Theory of Equations, p. 378.

It is evident that any covariant of a quantic, considered as a function of the roots and variables, is unchanged when any quantity is added to each root and variable. From this consideration with reference to coefficient functions, arose (1) and (2); and from the same consideration can be obtained the differential root equations corresponding to (1) and (2).

In any covariant function of the quantic

$$(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)\ldots, \tag{5}$$

or

$$(\alpha_1 \beta_1 \gamma_1 \ldots) y^n \left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right]\left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right]\left[\frac{x}{y} + \frac{\gamma_2}{\gamma_1}\right]\ldots, \tag{6}$$

let $\left[\frac{x}{y} + \lambda\right]$ be substituted for $\frac{x}{y}$, $\left[\frac{\alpha_2}{\alpha_1} - \lambda\right]$ for $\frac{\alpha_2}{\alpha_1}$, $\left[\frac{\beta_2}{\beta_1} - \lambda\right]$ for $\frac{\beta_2}{\beta_1}$, etc. Such a substitution leaves the covariant unchanged and is equivalent to the substitution

$$x = \bar{x} + \lambda\bar{y}, \quad y = 0 \, . \, \bar{x} + \bar{y};$$

$$\alpha_2 = \bar{\alpha}_2 - \lambda\bar{\alpha}_1, \quad \alpha_1 = 0 \, . \, \bar{\alpha}_2 + \bar{\alpha}_1; \text{ etc.} \tag{7}$$

Regarding the covariant as a covariant θ of the form of Art. 2, where

$$\theta = (\alpha_1 \beta_1 \gamma_1 \ldots)^w y^m \Sigma \left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{12}} \left[\frac{\gamma_2}{\gamma_1} - \frac{\beta_2}{\beta_1}\right]^{e_{23}} \ldots \left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right]^{e_1} \left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right]^{e_2} \ldots, \tag{8}$$

and letting

$$\Sigma = \theta\left[\frac{x}{y}, \frac{-\alpha_2}{\alpha_1}, \frac{-\beta_2}{\beta_1}, \frac{-\gamma_2}{\gamma_1}, \ldots\right] = \theta_0, \tag{9}$$

we have, by Taylor's Theorem,

$$\theta\left[\frac{x}{y} + \lambda, \frac{-\alpha_2}{\alpha_1} + \lambda, \frac{-\beta_2}{\beta_1} + \lambda, \ldots\right] = \theta_0 - \lambda\delta\theta_0 + \frac{\lambda^2}{1 \, . \, 2}\delta^2\theta_0 + \ldots, \tag{10}$$

where

$$\delta = \frac{d}{d\left[\frac{\alpha_2}{\alpha_1}\right]} + \frac{d}{d\left[\frac{\beta_2}{\beta_1}\right]} + \ldots - \frac{d}{d\left[\frac{x}{y}\right]}. \tag{11}$$

Since θ_0 is a covariant, and hence unchanged by the transformation (7), the coefficient of λ vanishes and therefore

$$\frac{d\theta_0}{d\left[\frac{\alpha_2}{\alpha_1}\right]} + \frac{d\theta_0}{d\left[\frac{\beta_2}{\beta_1}\right]} + \ldots - \frac{d\theta_0}{d\left[\frac{x}{y}\right]} = 0. \tag{12}$$

Regarding the quantic (5) as a function of $\frac{y}{x}$, and writing it in the form

$$(\alpha_2\beta_2\gamma_2 \ldots)\, x^n \left[\frac{y}{x} + \frac{\alpha_1}{\alpha_2}\right] \left[\frac{y}{x} + \frac{\beta_1}{\beta_2}\right] \left[\frac{y}{x} + \frac{\gamma_1}{\gamma_2}\right] \ldots, \tag{13}$$

θ may be written in the form

$$(\alpha_2\beta_2\gamma_2 \ldots)^{\omega}\, x^m\, \Sigma \left[\frac{\alpha_1}{\alpha_2} - \frac{\beta_1}{\beta_2}\right]^{e_{12}} \left[\frac{\beta_1}{\beta_2} - \frac{\gamma_1}{\gamma_2}\right]^{e_{23}} \ldots \left[\frac{y}{x} + \frac{\alpha_1}{\alpha_2}\right]^{e_1} \left[\frac{y}{x} + \frac{\beta_1}{\beta_2}\right]^{e_2} \ldots, \tag{14}$$

and is unchanged by the transformation

$$x = \bar{x} + 0 \,.\, \bar{y}, \qquad y = \lambda\bar{x} + \bar{y};$$

$$\alpha_2 = \bar{\alpha}_2 + 0 \,.\, \bar{\alpha}_1, \qquad \alpha_1 = -\lambda\bar{\alpha}_2 + \bar{\alpha}_1; \text{ etc.} \tag{15}$$

Now Σ in (14) can differ only in sign from

$$\theta\left[\frac{y}{x}, \frac{-\alpha_1}{\alpha_2}, \frac{-\beta_1}{\beta_2}, \frac{-\gamma_1}{\gamma_2}, \ldots\right];$$

also, by Taylor's Theorem,

$$\theta\left[\frac{y}{x} + \lambda, \frac{-\alpha_1}{\alpha_2} + \lambda, \frac{-\beta_1}{\beta_2} + \lambda, \ldots\right] = \theta_0 - \lambda\delta'\theta_0 + \frac{\lambda^2}{1\,.\,2}\delta'^2\theta_0 + \ldots, \tag{16}$$

where

$$\delta' = \frac{d}{d\left[\frac{\alpha_1}{\alpha_2}\right]} + \frac{d}{d\left[\frac{\beta_1}{\beta_2}\right]} + \ldots - \frac{d}{d\left[\frac{y}{x}\right]}, \tag{17}$$

and as before the coefficient of λ vanishes,

$$\therefore \quad \frac{d\theta_0}{d\left[\frac{\alpha_1}{\alpha_2}\right]} + \frac{d\theta_0}{d\left[\frac{\beta_1}{\beta_2}\right]} + \ldots - \frac{d\theta_0}{d\left[\frac{y}{x}\right]} = 0\,. \tag{18}$$

Operators (11) and (17) may be written

$$\delta = \alpha_1\frac{d}{d\alpha_2} + \beta_1\frac{d}{d\beta_2} + \ldots - y\frac{d}{dx}, \tag{19}$$

$$\delta' = \alpha_2\frac{d}{d\alpha_1} + \beta_2\frac{d}{d\beta_1} + \ldots - x\frac{d}{dy}. \tag{20}$$

Equations (12) and (18) may be written in the form

$$\alpha_1 \frac{d\theta}{d\alpha_2} + \beta_1 \frac{d\theta}{d\beta_2} + \ldots - y\frac{d\theta}{dx} = 0\,, \tag{22}$$

$$\alpha_2 \frac{d\theta}{d\alpha_1} + \beta_2 \frac{d\theta}{d\beta_1} + \ldots - x\frac{d\theta}{dy} = 0\,. \tag{23}$$

All root symbols of covariants, being the symbolic expressions of θ, satisfy both equations (22) and (23).

27. *Particular forms of the equations of Art.* 26. Operators (19) and (20) are adapted to the form of the general symbolic root expression. In the common method of root expression, when the quantic is

$$a_0(x - \alpha y)\,(x - \beta y)\,(x - \gamma y)\,\ldots\,, \tag{24}$$

the covariants and invariants are expressed as functions of a_0 and root differences. The quantic (24) is a particular form of the general quantic of the preceding sections

$$(\alpha_1 x + \alpha_2 y)\,(\beta_1 x + \beta_2 y)\,(\gamma_1 x + \gamma_2 y)\,\ldots\,,$$

when

$$\alpha_1\beta_1\gamma_1\ldots = a_0\,,\ \alpha_2 = -\alpha\alpha_1\,,\ \beta_2 = -\beta\beta_1\,,\ \gamma_2 = -\gamma\gamma_1\,,\ \ldots.$$

The form of operator (20) must be changed, in order to adapt it to the covariants and invariants of (24); operator (19) does not act upon $\alpha_1, \beta_1, \gamma_1, \ldots$, and suffers no alteration excepting the changes involved in the substitution of the values of $\alpha_2, \beta_2, \gamma_2, \ldots$.

According to Art. 2

$$\theta = \Sigma(\alpha_1\beta_2 - \alpha_2\beta_1)^{e_{12}}\,(\alpha_1\gamma_2 - \alpha_2\gamma_1)^{e_{13}}\ldots(\alpha_1 x + \alpha_2 y)^{e_1}\ldots \tag{25}$$

$$= (\alpha_1\beta_1\gamma_1\ldots)^{\omega}\,\Sigma\left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{12}}\left[\frac{\gamma_2}{\gamma_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{13}}\ldots\left[x + \frac{\alpha_2}{\alpha_1}y\right]^{e_1}\ldots \tag{26}$$

$$= a_0^{\omega}\,\Sigma\left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{12}}\left[\frac{\gamma_2}{\gamma_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{13}}\ldots\left[x + \frac{\alpha_2}{\alpha_1}y\right]^{e_1}\ldots \tag{27}$$

$$= \Sigma\, a_0^{e_{12}}\left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{12}}\cdot a_0^{e_{13}}\left[\frac{\gamma_2}{\gamma_1} - \frac{\alpha_2}{\alpha_1}\right]^{e_{13}}\ldots a_0^{e_1}\left[x + \frac{\alpha_2}{\alpha_1}y\right]^{e_1}\ldots \tag{28}$$

Hence,

$$\alpha_1\frac{d\theta}{d\alpha_1} + \alpha_2\frac{d\theta}{d\alpha_2} = \alpha_1\frac{d\theta}{da_0}\frac{da_0}{d\alpha_1} = a_0\frac{d\theta}{da_0}\,, \tag{29}$$

$$\alpha_1\alpha_2\frac{d\theta}{d\alpha_1} + \alpha_2^2\frac{d\theta}{d\alpha_2} = \alpha_2 a_0\frac{d\theta}{da_0}\,, \tag{30}$$

$$a_2\frac{d\theta}{da_1}+\frac{a_2^2}{a_1}\frac{d\theta}{da_2}=\frac{a_2}{a_1}a_0\frac{d\theta}{da_0}, \tag{31}$$

$$a_2\frac{d\theta}{da_1}=-\frac{a_2^2}{a_1}\frac{d\theta}{da_2}+\frac{a_2}{a_1}a_0\frac{d\theta}{da_0}. \tag{32}$$

Substituting in (23) the values of $a_2\frac{d\theta}{da_1}$, $\beta_2\frac{d\theta}{d\beta_1}$, . . . , gives

$$-\left[\frac{a_2^2}{a_1}\frac{d\theta}{da_2}+\frac{\beta_2^2}{\beta_1}\frac{d\theta}{d\beta_2}+\frac{\gamma_2^2}{\gamma_1}\frac{d\theta}{d\gamma_2}+\ldots\right] + \left[\frac{a_2}{a_1}+\frac{\beta_2}{\beta_1}+\frac{\gamma_2}{\gamma_1}+\ldots\right]a_0\frac{d\theta}{da_0}-x\frac{d\theta}{dy}=0, \tag{33}$$

and thus operator (20) becomes

$$\delta'=-\left[\frac{a_2^2}{a_1}\frac{d}{da_2}+\frac{\beta_2^2}{\beta_1}\frac{d}{d\beta_2}+\frac{\gamma_2^2}{\gamma_1}\frac{d}{d\gamma_2}+\ldots\right] + \left[\frac{a_2}{a_1}+\frac{\beta_2}{\beta_1}+\frac{\gamma_2}{\gamma_1}+\ldots\right]a_0\frac{d}{da_0}-x\frac{d}{dy}. \tag{34}$$

Substituting in (28) the values of a_2, β_2, γ_2 . . . which correspond to the form of quantic (24), the resulting value of θ is the general covariant for the form (24). Making the same substitutions in (19) and (34)

$$\delta=-\left[\frac{d}{da}+\frac{d}{d\beta}+\frac{d}{d\gamma}+\ldots+y\frac{d}{dx}\right], \tag{35}$$

$$\delta'=a^2\frac{d}{da}+\beta^2\frac{d}{d\beta}+\gamma^2\frac{d}{d\gamma}+\ldots-(a+\beta+\gamma+\ldots)\,a_0\frac{d}{da_0}-x\frac{d}{dy}, \tag{36}$$

where δ and δ' now act upon the forms belonging to the quantic (24), and correspond to the operators (19) and (20) of the general quantic. The differential equations corresponding to (35) and (36) are*

$$\frac{d\theta}{da}+\frac{d\theta}{d\beta}+\frac{d\theta}{d\gamma}+\ldots+y\frac{d\theta}{dx}=0, \tag{37}$$

$$a^2\frac{d\theta}{da}+\beta^2\frac{d\theta}{d\beta}+\gamma^2\frac{d\theta}{d\gamma}+\ldots-(a+\beta+\gamma+\ldots)\,a_0\frac{d\theta}{da_0}-x\frac{d\theta}{dy}=0. \tag{38}$$

If a_0 of (24) equal unity, and y equal unity, then (24) has the form

$$(x-a)(x-\beta)(x-\gamma)\ldots, \tag{39}$$

* Compare Salmon's M. H. Algebra, p. 65.

and the covariants and invariants of (39) satisfy the equations

$$\frac{d\theta}{d\alpha}+\frac{d\theta}{d\beta}+\frac{d\theta}{d\gamma}+\ldots+\frac{d\theta}{dx}=0\,, \tag{40}$$

$$\alpha^2\frac{d\theta}{d\alpha}+\beta^2\frac{d\theta}{d\beta}+\gamma^2\frac{d\theta}{d\gamma}+\ldots=0\,, \tag{41}$$

obtained from (37) and (38).

28. *Semi-invariants.* Semi-invariants are symmetric functions of the roots multiplied by $(\alpha_1\beta_1\gamma_1\ldots)^\omega$, and satisfying but one of the differential equations (22) and (23) of Art. 26. For example, the coefficient of x^2 in $\Sigma(\alpha\beta)^2\gamma^2$, which is $\Sigma(\alpha\beta)^2\gamma_1^{\,2}$ or $(\alpha_1\beta_1\gamma_1)^2\ \Sigma\left[\frac{\beta_2}{\beta_1}-\frac{\alpha_2}{\alpha_1}\right]^2$, is a semi-invariant: it satisfies (22), vanishing under the operator (19), but does not satisfy (23), nor vanish under the corresponding operator (20). Under the operator (20) $\Sigma(\alpha\beta)^2\gamma_1^{\,2}$ produces $2\Sigma(\alpha\beta)^2\gamma_1\gamma_2$, which is the coefficient of xy in $\Sigma(\alpha\beta)^2\gamma^2$; and upon the application of (20), $\Sigma(\alpha\beta)^2\gamma_1\gamma_2$ produces $\Sigma(\alpha\beta)^2\gamma_2^{\,2}$, which is the coefficient of y^2 in $\Sigma(\alpha\beta)^2\gamma^2$, and a semi-invariant. This last coefficient satisfies (23) but not (22), and under the action of operator (19) produces the preceding coefficients.

In general the source* of every covariant is a semi-invariant which satisfies (22); and the operator (20) applied to the source produces in succession the remaining coefficients of the given covariant. The final coefficient is a semi-invariant which satisfies (23) and under the action of the operator (19) gives rise to the preceding coefficients in the covariant.

It may be remarked that a semi-invariant is either a symmetric function of the differences of the roots multiplied by $(\alpha_1\beta_1\gamma_1\ldots)^\omega$, or a symmetric function of the differences of the reciprocals of the roots multiplied by $(\alpha_2\beta_2\gamma_2\ldots)^\omega$. All semi-invariants which are sources of covariants belong to the first class; all semi-invariants which are final coefficients belong to the second class.

29. *Semi-invariants of the quantic* $(x-\alpha)(x-\beta)(x-\gamma)\ldots$. The root forms of the semi-invariants of the quantic $(x-\alpha)(x-\beta)(x-\gamma)\ldots$ are simple and interesting in form. Here the covariant whose symbol is $\Sigma(\alpha\beta)^2\gamma^2$ has the form $\Sigma(\alpha-\beta)^2(x-\gamma)^2$ and its source is the semi-invariant $\Sigma(\alpha-\beta)^2$, and its final coefficient the semi-invariant $\Sigma(\alpha-\beta)^2\gamma^2$. In any covariant symbol, the determinant part, expressed as a function of root differences, is the root expression for the semi-invariant, which is the source of the given covariant. It should be noticed that semi-invariants of the quantic† $(x-\alpha)(x-\beta)$

* The *source* is the coefficient of the highest power of x in the covariant. See Salmon, p. 134; M. Roberts, Quart. Journ. Vol. IV.

† Concomitants of the quantic $a_0(x-\alpha)(x-\beta)(x-\gamma)\ldots$ are obtained from those of the quantic $(x-\alpha)(x-\beta)(x-\gamma)\ldots$ by multiplying by $a_0{}^\omega$.

$(x-\gamma)\ldots$ are symmetric functions of roots and root differences, which either do not involve all of the roots of the quantic, or if all the roots occur, do not involve them similarly; semi-invariants might be defined as such functions of the roots. Those semi-invariants which are sources are functions of root differences only, as in the above example.

If any semi-invariant summation of root differences be given, the covariant of which it is the source can be formed directly; let $\Sigma(\alpha-\beta)(\gamma-\delta)$ and $\Sigma(\alpha-\beta)^3(\gamma-\delta)^2(\alpha-\gamma)$ be semi-invariants of the quintic; then it is evident that $\Sigma(\alpha-\beta)(\gamma-\delta)(x-\varepsilon)$ is the covariant of which the first is the source, and that $\Sigma(\alpha-\beta)^3(\gamma-\delta)^2(\alpha-\gamma)(x-\beta)(x-\gamma)(x-\delta)^2(x-\varepsilon)^4$ is the covariant of which the second is the source.

Here the covariant root summation is obtained from its source by annexing such quantic factors as are sufficient to cause each root of the quantic to appear as many times as the highest number of times any one root appears in the semi-invariant summation which is the source. This connection between covariants and their sources in the root forms is an attractive feature of the root functions expressed as symmetric functions of root differences.

A peculiarity of the final semi-invariant in a covariant in the root difference form, is worthy of notice; if in the final coefficient $(x-\alpha)$ be substituted for α, $(x-\beta)$ for β, etc., there results the covariant of which the given semi-invariant is the final coefficient; thus making these substitutions in $\Sigma(\alpha-\beta)^2\gamma^2$, there results $\Sigma(\alpha-\beta)^2(x-\gamma)^2$. Similarly* if in the source of a covariant $\dfrac{1}{x-\alpha}$ be substituted for α, $\dfrac{1}{x-\beta}$ for β, etc., and the result multiplied by $(x-\alpha)^\omega(x-\beta)^\omega(x-\gamma)^\omega\ldots$, the covariant with the given source is obtained.

30. *Invariant functions of the* $n$ic *become semi-invariants of the* $(n+k)$ic. All summations of root differences that are invariants for any quantic become semi-invariants of higher quantics when the summation is applied to the roots of higher quantics; but only those semi-invariants of the higher quantic, which involve roots similarly, have corresponding invariants in some lower quantic. Thus $\Sigma(\alpha-\beta)^2$ of a quadratic is an invariant; but $\Sigma(\alpha-\beta)^2$ of a higher quantic is the semi-invariant which is the source of the covariant $\Sigma(\alpha-\beta)^2(x-\gamma)^2(x-\delta)^2\ldots$, which is H of the $(2+k)$ic, and whose root symbol is $\Sigma(\alpha\beta)^2\gamma^2\delta^2\ldots.$ The semi-invariant $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)$, which is the source of the covariant whose symbol is $\Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\gamma)\beta\gamma\delta^2\varepsilon^4$, cannot be an invariant of any quantic. It is evident also that any semi-invariant of the $n$ic is a semi-invariant of the $(n+k)$ic.

31. *Fundamental semi-invariants.* The following root expressions are the root forms of those semi-invariants, which have been shown by Cayley and

* Burnside and Panton, Theory of Equations, p. 366.

others to be the fundamental semi-invariants in terms of which all semi-invariants of the *n*ic can be rationally and integrally expressed :*

Root Form	Coefficient Form	Source of
$\alpha_1\beta_1\gamma_1 \ldots$	a_0	$a_0x^n + \ldots$
$\Sigma(\alpha\beta)^2(\alpha\gamma)\beta_1\gamma_1^2$	$a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$	$\overline{12}^2 . \overline{13}$
$\Sigma(\alpha\beta)^2(\alpha\gamma)(\alpha\delta)\beta_1^2\gamma_1^3\delta_1^3$	$a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0^2a_2^2 - 3a_1^4$	$\overline{12}^2 . \overline{13} . \overline{14}$
$\Sigma(\alpha\beta)^2(\alpha\gamma)(\alpha\delta)(\alpha\varepsilon)\beta_1^3\gamma_1^4\delta_1^4\varepsilon_1^4$	$a_0^4a_5 - 5a_0^3a_1a_4 + 10a_0^2a_1^2a_3 - 10a_0a_1^3a_2 + 4a_1^5$	$\overline{12}^2 . \overline{13} . \overline{14} . \overline{15}$
...	...	...

In these the coefficient forms are of the type $a_0^{-1}(a_0, a_1, a_2, \ldots a_m)(-a_1, a_0)^m$.*

32. *Semi-covariants.*† Semi-covariants are functions of the roots and variables which satisfy but one of the pair of differential equations (22) and (23) of Art. 26. E. g. if the summation sign of any covariant symmetric function of the roots and variables of an *n*ic, be extended to the roots of an $(n+k)$ic there results a semi-covariant of the $(n+k)$ic. All semi-covariants of an *n*ic are products of $(\alpha_1\beta_1\gamma_1 \ldots)^w y^m$ and symmetric functions of the differences of the roots and variables of the *n*ic; into which functions either all the roots of the *n*ic do not enter, or if they all enter, are not similarly involved; thus for the quartic $(\alpha_1\beta_1\gamma_1\delta_1)^2 \;\Sigma \left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^2 \left[\frac{x}{y} + \frac{\gamma_2}{\gamma_1}\right]^2$ and $(\alpha_1\beta_1\gamma_1\delta_1)^2 \;\Sigma \left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^2 \left[\frac{x}{y} + \frac{\gamma_2}{\gamma_1}\right] \left[\frac{x}{y} + \frac{\delta_2}{\delta_1}\right]$ are semi-covariants. These may be written in the forms $\Sigma(\alpha\beta)^2\gamma^2\delta_1^2$ and $\Sigma(\alpha\beta)^2\gamma\delta\gamma_1\delta_1$; and for the quantic $a_0(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ they become $a_0^2\,\Sigma(\alpha-\beta)^2(x-\gamma)^2$ and $a_0^2\,\Sigma(\alpha-\beta)^2(x-\gamma)(x-\delta)$.

If, in the final coefficient, $\left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right]$ be substituted for $\frac{\alpha_2}{\alpha_1}$, $\left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right]$ for $\frac{\beta_2}{\beta_1}$, etc., there results the semi-covariant itself; e. g. $\Sigma(\alpha\beta)^2\alpha_1^2\beta_1^2\gamma_1^4\delta_2^4$ is equal to $(\alpha_1\beta_1\gamma_1\delta_1)^4 \;\Sigma \left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^2 \left[\frac{\delta_2}{\delta_1}\right]^4$, which becomes $(\alpha_1\beta_1\gamma_1\delta_1)^4\,\Sigma \left[\frac{\beta_2}{\beta_1} - \frac{\alpha_2}{\alpha_1}\right]^2 \left[\frac{x}{y} + \frac{\delta_2}{\delta_1}\right]^4$ when the substitutions are made. The final coefficient under the successive operation of δ produces the root expressions for the preceding coefficients.

* Cayley, *Am. Jour.*, Vol. VIII, p. 59; A. B. Kempe, *London Math. Soc.* (1893), p. 105; E. B. Elliott, *Messenger of Math.*, Vol. XXIII, p. 91.

† Burnside and Panton.

There are fundamental semi-covariants in terms of which all semi-covariants can be expressed as rational integral functions. Let $S =$ any semi-covariant, and $s =$ the final coefficient in S; and let φ and θ be rational integral functions which involve the roots in the same degree. Then

$$s = (\alpha_1 \beta_1 \ldots)^m \varphi \left[\frac{\alpha_2}{\alpha_1}, \frac{\beta_2}{\beta_1}, \ldots \right]$$

$$= a_0^m \theta \left[\frac{a_1}{a_0}, \frac{a_2}{a_0}, \ldots \right]$$

$$= a_0^m \theta \left[\Sigma \frac{\alpha_2}{\alpha_1}, \Sigma \frac{\alpha_2 \beta_2}{\alpha_1 \beta_1}, \ldots \right].$$

Substituting $\left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right]$ for $\frac{\alpha_2}{\alpha_1}$, $\left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right]$ for $\frac{\beta_2}{\beta_1}$, etc., in s, we obtain S; therefore

$$S = a_0^m \theta \left[\Sigma \left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right], \ \Sigma \left[\frac{x}{y} + \frac{\alpha_2}{\alpha_1}\right] \left[\frac{x}{y} + \frac{\beta_2}{\beta_1}\right], \ldots \right] \tag{4}$$

$$= y^m \theta \left[\Sigma(\alpha_1 x + \alpha_2 y), \ \Sigma(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y), \ldots \right]; \tag{5}$$

hence $\Sigma(\alpha_1 x + \alpha_2 y)$, $\Sigma(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)$, etc., may be taken as fundamental semi-covariants in terms of which any semi-covariant S can be expressed as a rational integral function. The series of fundamental semi-covariants are the 1st, 2nd, ..., $(n-1)$th derivatives of the $n$ic, $(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y) \ldots$.

If the $n$ic be $a_0 (x - \alpha)(x - \beta)(x - \gamma) \ldots$, then (4) becomes

$$S = a_0^m \theta \left[\Sigma(x - \alpha), \ \Sigma(x - \alpha)(x - \beta), \ldots \right].$$

33. *Semi-invariants and semi-covariants of a system of quantics.* What has been said in Articles 28–32 concerning certain concomitants of a single binary quantic, can be extended to similar concomitants of a system of quantics, where the operators and differential equations are obtained from those of Art. 26 by annexing similar root terms for each quantic of the system. Thus (19) and (20) of Art. 26 become for a system of quantics

$$\delta = \alpha_1 \frac{d}{d\alpha_2} + \beta_1 \frac{d}{d\beta_2} + \ldots + \alpha'_1 \frac{d}{d\alpha'_2} + \beta'_1 \frac{d}{d\beta'_2} + \ldots - y \frac{d}{dx}, \tag{1}$$

$$\delta' = \alpha_2 \frac{d}{d\alpha_1} + \beta_2 \frac{d}{d\beta_1} + \ldots + \alpha'_2 \frac{d}{d\alpha'_1} + \beta'_2 \frac{d}{d\beta'_1} + \ldots - x \frac{d}{dy}. \tag{2}$$

It is evident that $\Sigma(\alpha\alpha')(\beta\beta')$ is a semi-invariant of any system of 2ic and $(2+k)$ic, or of $(2+h)$ic and $(2+k)$ic, and is the source of $\Sigma(\alpha\alpha')(\beta\beta')\gamma\gamma'\delta\delta' \ldots$;

and that $\Sigma(aa')(\beta\beta')\gamma\gamma'$ is a semi-covariant of a system of 3ic and $(3+k)$ic, or of $(3+h)$ic and $(3+k)$ic.

34. *Intermediate forms.* If f and f' be quantics of the same degree, and if a covariant of the quantic $f+\lambda f'$ be formed, the coefficients of the successive powers of λ in that covariant, are covariants of the system of f and f'. These coefficients have been called the *intermediate* covariants of the system f, f'; and in coefficient forms (not symbolic) the operator*

$$a'\frac{d}{da}+b'\frac{b}{db}+c'\frac{d}{dc}+\dots \tag{1}$$

has been used by Boole and others to obtain these forms from the corresponding covariants of the single quantic f. Let (1) be written in the form

$$\bar{a}_0'\frac{d}{da_0}+\bar{a}_1'\frac{d}{da_1}+\bar{a}_2'\frac{d}{da_2}+\dots \tag{2}$$

where

$$f=\bar{a}_0x^n+n\bar{a}_1x^{n-1}y+\dots+\frac{n!}{(n-r)!\,r!}\bar{a}_rx^{n-r}y^r+\dots;$$

also let f be written in the form

$$\begin{aligned} f &= (a_1x+a_2y)^n \\ &= a_1^nx^n+na_1^{n-1}a_2x^{n-1}y+\dots+\frac{n!}{(n-r)!\,r!}a_1^{n-r}a_2^rx^{n-r}y^r+\dots; \end{aligned}$$

then

$$\bar{a}_r=a_1^{n-r}a_2^r,$$

$$\frac{d}{da_r}=\frac{1}{(n-r)!\,r!}\left[\frac{d}{da_1}\right]^{n-r}\left[\frac{d}{da_2}\right]^r.$$

When these substitutions are made in (2), with a corresponding substitution for $\bar{a}'_r$, the operator (2) becomes

$$\frac{1}{n!}\left\{a_1'^n\left[\frac{d}{da_1}\right]^n+na_1'^{n-1}a_2'\left[\frac{d}{da_1}\right]^{n-1}\left[\frac{d}{da_2}\right]+\dots+\frac{n!}{(n-r)!\,r!}a_1'^{n-r}a_2'^r\left[\frac{d}{da_1}\right]^{n-r}\left[\frac{d}{da_2}\right]^r+\dots\right\}, \tag{2}$$

and may be written in the form

$$\frac{1}{n!}\left[a_1'\frac{d}{da_1}+a_2'\frac{d}{da_2}\right]^n, \tag{3}$$

* Boole, *Cam. Math. Jour.* (1841), Vol. III, p. 106. Salmon's M. H. Algebra, p. 211. Gordan's Vorlesungen, Vol. II, p. 60.

which may be represented by $\left[a' \frac{d}{da}\right]^n$. If (3) be applied to the coefficient symbol of any covariant, the corresponding intermediate covariants may be obtained. Thus to find the intermediate covariants of $(ab)^2ab$ in a system of two cubics

$$\left[a' \frac{d}{da}\right] ((ab)^2ab) = 2(ab)(a'b)ab = A_1, \tag{4}$$

$$\left[a' \frac{d}{da}\right] A_1 = 2(a'b)^2 ab = A_2, \tag{5}$$

$$\left[a' \frac{d}{da}\right] A_2 = 2(a'b)^2 a'b, \tag{6}$$

$$\therefore \quad \left[a' \frac{d}{da}\right]^n ((ab)^2ab) = 2(a'b)^2 a'b, \tag{7}$$

and $(a'b)^2 a'b$ is the symbolic coefficient form of the intermediate H of two cubics (see Table IX, No. 2). A second application of (3) to $(ab)^2ab$ gives $(a'b')^2a'b'$, which is H of the cubic f'.

An operator of the form $\left[\alpha_1' \frac{d}{d\alpha_1} + \alpha_2' \frac{d}{d\alpha_2}\right] \left[\beta_1' \frac{d}{d\beta_1} + \beta_2' \frac{d}{d\beta_2}\right] \ldots$, which may be denoted by

$$\left[\alpha' \frac{d}{d\alpha}\right] \left[\beta' \frac{d}{d\beta}\right] \ldots, \tag{8}$$

applied to the root symbol of any covariant or invariant, produces other root symbols which correspond in order, weight, and degree with those of the intermediate forms.

35. *Application of the intermediate root operator.* To illustrate the action of the operator (8), Art. 34, let it be applied to D of the cubic.

$$D = (\alpha\beta)^2 (\alpha\gamma)^2 (\beta\gamma)^2, \tag{1}$$

$$\left[\alpha' \frac{d}{d\alpha}\right] D = 2\Sigma(\alpha'\beta)(\alpha\beta)(\alpha\gamma)^2(\beta\gamma)^2 = A_1, \tag{2}$$

$$\left[\beta' \frac{d}{d\beta}\right] A_1 = 2\Sigma(\alpha'\beta)(\alpha\beta')(\alpha\gamma)^2(\beta\gamma)^2 = A_2, \tag{3}$$

$$\left[\gamma' \frac{d}{d\gamma}\right] A_2 = 4\Sigma(\alpha'\beta)(\alpha\beta')(\alpha\gamma')(\alpha\gamma)(\beta\gamma)^2, \tag{4}$$

$$\therefore \quad \left[\alpha' \frac{d}{d\alpha}\right] \left[\beta' \frac{d}{d\beta}\right] \left[\gamma' \frac{d}{d\gamma}\right] D = \Sigma(\alpha'\beta)(\alpha\beta')(\alpha\gamma')(\alpha\gamma)(\beta\gamma)^2, \tag{5}$$

which is M of the system of two cubics ; and may also be written, as in Table IX, No. 10 :

$$M = \Sigma (\alpha\,{}_{\prime}\beta)^2 (\alpha\gamma)(\gamma\alpha')({}_{\prime}\beta\beta')(\gamma\gamma') \,.$$

Similarly, the same operator applied to M produces $\Sigma(\alpha'\,{}_{\prime}\beta')^2(\alpha\gamma')^2({}_{\prime}\beta\gamma)^2$, which is N of two cubics ; and applied to N produces $\Sigma(\alpha'\,{}_{\prime}\beta')^2(\alpha'\gamma)(\alpha\gamma')({}_{\prime}\beta'\gamma)({}_{\prime}\beta\gamma')$, which is M' of two cubics. A fourth application of this operator reduces D to D' of the quantic f'.

PART II.—SOME GEOMETRICAL INTERPRETATIONS AND APPLICATIONS.

CHAPTER I.

GEOMETRY OF BINARY FORMS.

36. *The relation of the binary nic to two ternary quantics.* A binary $n$ic equated to zero is an equation in x/y, whose roots are particular values of this ratio. It is evident that the intersection of two curves, represented by ternary equations, give points which satisfy the binary equation, obtained by eliminating one variable (z) between the two ternary equations which represent the curves. Thus, we may regard a binary $n$ic as an expression denoting the n points of intersection of two curves of the degrees μ and ν, where $\mu\nu = n$; the simplest and most direct interpretation being that in which $\mu = n$ and $\nu = 1$. This last interpretation, with the line $z = 0$ as the line ν, is directly applicable to the ternary and binary $n$ic in their most general forms; since the binary is the ternary form in which $z = 0$. Invariants and covariants of a binary thus interpreted, are invariants and covariants of the system of ternary curve and line.* Mr. Burnside† has used the case $\nu = 2$, $\mu = \frac{1}{2}n$, where the intersections of the two curves are the roots of a binary of even degree; and the case $\nu = 2$, $\mu = n$, where the points of tangency of the curve μ with the conic ν are the roots of a binary of odd degree. To obtain convenient forms for this interpretation, the following transformation is used:

$$x^2 = \bar{x}, \quad 2xy = \bar{y}, \quad y^2 = \bar{z}$$

and the binary quadratic is represented by the intersection of a line with the conic $4\bar{x}\bar{z} - \bar{y}^2 = 0$; and any binary $n$ic denotes n points on this conic.

37. *The relation of the binary nic to k-ary quantics.* The representation of the binary $n$ic by the intersection of two ternary loci, is a particular case of its representation by the intersections of a system of $k - 1$ loci defined by k-ary equations, in space of $k - 1$ dimensions. The case $k = 4$ has been developed by Lindemann and others.‡ In this case a binary $n$ic denotes the

* Salmon's M. H. Algebra, p. 172.

† Burnside and Panton, Theory of Equations (3d ed.), p. 455. Burnside, Quarterly Journal, Vol. X, p. 211. Salmon's M. H. Algebra, pp. 173, 181, etc.

‡ F. Lindemann, *Math. Ann.*, Vol. XXIII. W. R. W. Roberts, *Proc. London Math. Soc.*, Vol. XIII.

points of intersection of a system of three surfaces of degrees λ, μ, ν, where $\lambda\mu\nu = n$; and the general binary $n$ic may be represented by the points of intersection of the general quaternary $n$ic with the planes $z = 0$ and $w = 0$.

Mr. W. R. W. Roberts in a paper on two cubics, employed the transformation

$$\bar{x} = x^3, \quad \bar{y} = x^2y, \quad \bar{z} = xy^2, \quad w = y^3,$$

by which the binary cubic is represented by a plane that cuts the twisted cubic in three points; and all binary quantics denote n points on the twisted cubic.

38. *The system of ternary* $n$ic *and line* $z = 0$. We now return to a more detailed consideration of the system of line and curve as the geometrical exposition of the binary; it is in this view that the root forms of the general binary quantic find their most natural and most direct interpretation. The coefficients and roots of the binary $n$ic will be regarded as general complex numbers; so that all roots may be conceived to be on the complex plane, the axis of real quantities being the line $z = 0$, upon which lie the real roots of the $n$ic.

39. *The roots of the binary quantic.* The binary quantic vanishes for particular values of the ratio x/y. To obtain a definite geometrical idea of this ratio, let us consider the position of points on the complex plane with reference to two fixed base points A, B. All distances will be regarded as vector distances, measured by complex numbers, which may become real or purely imaginary. The distance of any point from A will be understood to be equal to the vector sum of the distances measured, on the axes of real and imaginary quantities, by the complex number that determines the affix of that point, the point A being the origin; and similarly for the distance of any point from B. The difference of the distances of the point from A and from B is the vector AB; and the differences of the complex numbers that measure these distances, is a real number, and measures the distance AB.

Let the distance of any point r from A, B, be a_r, b_r, and let

$$a_r = \rho x_r, \quad b_r = -\rho y_r,$$

then

$$\frac{a_r}{b_r} = -\frac{x_r}{y_r}, \quad a_r - b_r = AB,$$

and the point r may be taken to represent the root $-x_r/y_r$ or the equation

$$xy_r + yx_r = 0.$$

Similarly, the equation

$$(xy_1 + x_1y)(xy_2 + x_2y)\ldots(xy_n + x_ny) = 0$$

is represented by n points on the complex plane.

40. *The geometrical meaning of* $x_r y_s - x_s y_r$. Let AB be the unit of measure of vector distance, and let the respective vector-pairs of any two points r, s be determined by

$$\frac{a_r}{b_r} = -\frac{x_r}{y_r}, \quad \frac{a_s}{b_s} = -\frac{x_s}{y_s},$$

$$a_r - b_r = a_s - b_s = AB = 1\,;$$

then

$$\frac{a_r}{a_r - b_r} = \frac{x_r}{x_r + y_r}, \quad \frac{a_s}{a_s - b_s} = \frac{x_s}{x_s + y_s},$$

$\therefore$ the measures of the vectors a_r, a_s, are

$$\frac{x_r}{x_r + y_r}, \quad \frac{x_s}{x_s + y_s}$$

and the measure of the vector rs is

$$\frac{x_s}{x_s + y_s} - \frac{x_r}{x_r + y_r} = \frac{x_s y_r - x_r y_s}{(x_r + y_r)(x_s + y_s)}$$

in terms of the unit vector AB.

The latter fraction, when multiplied by any number, expresses the distance rs in terms of a new unit of measure, which equals the original unit divided by the multiplier of the fraction. Let the fraction be multiplied by $-(x_r + y_r)$ $(x_s + y_s)$; then the distance rs is expressed by $x_r y_s - x_s y_r$ in terms of a new unit vector k, when

$$k = \frac{-AB}{(x_r + y_r)(x_s + y_s)} = \frac{-\rho^2 AB}{(a_r - b_r)(a_s - b_s)} = -\rho^2 AB\,.$$

If there be a product

$$(x_1 y_2 - x_2 y_1)^{e_{12}} (x_1 y_3 - x_3 y_1)^{e_{13}} \ldots (x_m y_n - x_n y_m)^{e_{mn}}$$

each factor is the distance between two points, and all in terms of the same unit of measure.

41. *Root notation.* In the preceding sections x_r/y_r is replaced by a_1/a_2, x_2/y_2 by β_1/β_2, x_3/y_3 by γ_1/γ_2, etc.; and the quantic appears in the form

$$(a_1 x + a_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y) \ldots$$

which is often written in the abbreviated form $a\beta\gamma \ldots$.

In the following sections, for the sake of convenience, the quantic is most frequently written

$$(x - a)(x - \beta)(x - \gamma) \ldots$$

and the numbers $\alpha, \beta, \gamma, \ldots$, are represented on the complex plane by vector distances measured from a single base point A.

The root expressions of preceding sections are transferred into this notation by substituting $\alpha - \beta$ for $\alpha_1\beta_2 - \alpha_2\beta_1$, which is written $(\alpha\beta)$ in the Tables; and $x - \alpha$ for $\alpha_1 x + \alpha_2 y$, which is written α in the Tables.

42. *Covariants and invariants.* It follows from Arts. 2, 3, 40 that covariants and invariants are functions of the distances between points on the complex plane, which, when equated to zero, express relations among these points, that are unchanged by any linear transformation, and therefore by any change of base points. An invariant equated to zero expresses a relation among the n points denoted by the quantic. A covariant of degree p is represented by its p root points, that bear certain relations to the n root points of the quantic.

CHAPTER II.

PARTICULAR INVARIANTS AND COVARIANTS WITH GEOMETRICAL INTERPRETATIONS.

43. The following interpretations of the root expressions of certain invariants and covariants are given to illustrate the general geometrical theory adopted in the last section; and also to emphasize the importance of the root expressions in furnishing direct solutions and interpretations of well known forms. The geometrical problems considered are approached only through the root expressions. No attempt is made to investigate the elaborate forms of the quintic and sextic, since such an investigation would necessarily involve long and intricate algebraic work and add nothing to the general theory and methods, which are sufficiently illustrated in the forms of the lower quantics. Frequent reference is made to a paper by Mr. R. Russell (*Proc. London Math. Soc.*, Vol. XIX, p. 56) in which the irreducible invariant and covariant relations of quartic roots have been fully developed.

44. *H of the cubic.* Equating H to zero

$$(\alpha-\beta)^2(x-\gamma)^2+(\alpha-\gamma)^2(x-\beta)^2+(\beta-\gamma)^2(x-\alpha)^2=0\,, \tag{1}$$

$$\therefore \quad \frac{(\alpha-\beta)^2(x-\gamma)^2}{(\alpha-\gamma)^2(x-\beta)^2}+1+\frac{(\beta-\gamma)^2(x-\alpha)^2}{(\alpha-\gamma)^2(x-\beta)^2}=0\,, \tag{2}$$

i. e. $$\lambda^2+1+(1-\lambda)^2=0\,,\ \text{or}\ \lambda^2-\lambda+1=0\,, \tag{3}$$

where $$\lambda=\frac{(\alpha-\beta)(x-\gamma)}{(\alpha-\gamma)(x-\beta)}=\{\alpha\beta x\gamma\}\,. \tag{4}$$

Equation (3) is the condition that four points form an equi-anharmonic series. The vanishing of H, therefore, expresses the condition that the cubic points

and either one of the H points form an equi-anharmonic series ; and the cubic points are three of the vertices of a pair of equi-anharmonic quadrangles.* By solving the quadratic (3), we obtain the two linear factors of H:

$$H = [(\alpha + \omega\gamma + \omega^2\beta)\,x + (\beta\gamma + \omega\alpha\beta + \omega^2\alpha\gamma)]$$
$$[(\alpha + \omega^2\gamma + \omega\beta)\,x + (\beta\gamma + \omega^2\alpha\beta + \omega\alpha\gamma)]$$

where $\omega^3 = 1$. By changing the form of the factors the H equation may be written

$$\left\{\frac{1}{x-\alpha} + \frac{\omega}{x-\beta} + \frac{\omega^2}{x-\gamma}\right\}\left\{\frac{1}{x-\alpha} + \frac{\omega^2}{x-\beta} + \frac{\omega}{x-\gamma}\right\} = 0\,, \tag{5}$$

and also

$$\left\{\frac{1}{x-\alpha} - \frac{1}{x-\gamma} + \omega\left[\frac{1}{x-\beta} - \frac{1}{x-\gamma}\right]\right\}$$
$$\left\{\frac{1}{x-\alpha} - \frac{1}{x-\beta} + \omega\left[\frac{1}{x-\gamma} - \frac{1}{x-\beta}\right]\right\} = 0\,. \tag{6}$$

Therefore, the H points must satisfy the equations

$$\left[\frac{1}{x-\alpha} - \frac{1}{x-\gamma}\right] + \omega\left[\frac{1}{x-\beta} - \frac{1}{x-\gamma}\right] = 0\,, \tag{7}$$

$$\left[\frac{1}{x-\alpha} - \frac{1}{x-\beta}\right] + \omega\left[\frac{1}{x-\gamma} - \frac{1}{x-\beta}\right] = 0\,. \tag{8}$$

From (7) it is evident the line joining the quasi-inverses† of the points α and γ is inclined at an angle of 60° (= argument of ω) to the line joining the quasi-inverses of the points γ and β, when x (one of the H points) is the point of inversion ;‡ similarly for the points (α, β) and (β, γ) of equation (8). It is now evident that the H points are points from which the triangle α, β, γ inverts into an equilateral triangle.

If $\alpha = \beta$ in the cubic ; that is, if $D = 0$, which is

$$(\alpha - \beta)^2\,(\alpha - \gamma)^2\,(\beta - \gamma)^2 = 0\,,$$

then (1) becomes

$$(x - \alpha)^2\,(\alpha - \gamma)^2 = 0\,, \tag{9}$$

* For the construction of the equi-anharmonic quadrangle, see Harkness and Morley, *Theory of Functions*, page 25.

† Quasi-inversion is circle inversion from a point, and a reflection about the axis of real quantities.

‡ This Geometrical interpretation of H is due to Mr. R. Russell. See *Proc. London Math. Soc.*, Vol. XIX, p. 62.

and H has a square factor, the same square factor that occurs in the cubic.*

If $\alpha = \beta = \gamma$, equation (1) is true identically.

45. *The covariant J of the cubic.*† Equating $\Sigma(\alpha\beta)^2(\beta\gamma)\alpha\gamma^2$ to zero, and dividing by $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma)$, we get

$$-\lambda + \frac{1}{\lambda} - \left[1 - \frac{1}{\lambda}\right] + \frac{\lambda}{\lambda-1} - \frac{1}{1-\lambda} + (1-\lambda) = 0\,,$$

$$\therefore\quad 2\lambda^3 - 3\lambda^2 + 3\lambda + 2\,, = (\lambda+1)(2\lambda-1)(\lambda-2)\,, = 0\,,$$

$\therefore \lambda = -1, \frac{1}{2}, 2$; and any one of the roots of J is harmonic with the three roots of the cubic.‡

46. *S of the quartic.* Equating S, $\Sigma(\alpha\beta)^2(\gamma\delta)^2$, to zero,

$$(\alpha-\beta)^2(\gamma-\delta)^2 + (\beta-\gamma)^2(\delta-\alpha)^2 + (\alpha-\gamma)^2(\beta-\delta)^2 = 0\,, \tag{1}$$

$$\therefore\quad \frac{(\alpha-\beta)^2(\gamma-\delta)^2}{(\alpha-\delta)^2(\beta-\gamma)^2} + \frac{(\alpha-\gamma)^2(\beta-\delta)^2}{(\beta-\gamma)^2(\delta-\alpha)^2} + 1 = 0\,, \tag{2}$$

i. e.

$$\{\alpha\beta\gamma\delta\}^2 + \{\alpha\gamma\beta\delta\}^2 + 1 = 0\,, \tag{3}$$

$$\therefore\quad \lambda^2 + (1-\lambda)^2 + 1 = 0\,,\quad [\lambda = \{\alpha\beta\gamma\delta\}] \tag{4}$$

$$\therefore\quad \lambda^2 - \lambda + 1 = 0\,. \tag{5}$$

Therefore $\alpha, \beta, \gamma, \delta$ are in equi-anharmonic ratio (see Art. 44).

47. *T of the quartic.* Equating T, $= \Sigma(\alpha\beta)^2(\gamma\delta)^2(\alpha\gamma)(\beta\delta)$, to zero and dividing by $(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\gamma-\delta)(\delta-\beta)$ we obtain

$$\lambda - \frac{1}{\lambda} - \left[\frac{\lambda}{\lambda-1}\right] - (1-\lambda) + \frac{1}{1-\lambda} + \left[\frac{\lambda}{\lambda-1}\right] = 0\,,$$

$$\therefore\quad (\lambda+1)(2\lambda-1)(\lambda-2) = 0\,,$$

$$\therefore\quad \lambda = -1\,,\ \tfrac{1}{2},\ 2\,.$$

Hence the vanishing of T is the condition that the four roots may be harmonic.

48. *J of the quartic.* Equating J, $= \Sigma(\alpha\beta)^2(\gamma\alpha)\gamma^2\beta\delta^3$, to zero, and factoring,

$$\begin{aligned}&[(x-\gamma)(x-\alpha)(\beta-\delta) - (x-\beta)(x-\delta)(\alpha-\gamma)]\\ \times&[(x-\alpha)(x-\beta)(\gamma-\delta) + (x-\gamma)(x-\delta)(\alpha-\beta)]\\ \times&[(x-\beta)(x-\delta)(\alpha-\gamma) + (x-\alpha)(x-\gamma)(\beta-\delta)] = 0\end{aligned} \tag{1}$$

* This is a particular case of the following theorem, which is evident from the root-forms: If a binary quantic have a square factor, this is a factor of the Hessian.

† J is the evectant of D. See Art. 24.

‡ For the construction of the harmonic quadrangle see Harkness and Morley, p. 24.

which may be written

$$\begin{vmatrix} x^2 & 2x & 1 \\ \beta\gamma & \beta+\gamma & 1 \\ \alpha\delta & \alpha+\delta & 1 \end{vmatrix} \begin{vmatrix} x^2 & 2x & 1 \\ \alpha\gamma & \alpha+\gamma & 1 \\ \beta\delta & \beta+\delta & 1 \end{vmatrix} \begin{vmatrix} x^2 & 2x & 1 \\ \alpha\beta & \alpha+\beta & 1 \\ \gamma\delta & \gamma+\delta & 1 \end{vmatrix} = 0 . \qquad (2)$$

The vanishing of a factor in (2) expresses that a pair of J points are the double points of the involution formed by the corresponding pairs of quartic points, the distances α, β, γ, δ being measured along an axis. Equation (1) may be put into the form

$$\left\{\frac{1}{x-\gamma}+\frac{1}{x-\alpha}-\frac{1}{x-\beta}-\frac{1}{x-\delta}\right\}\left\{\frac{1}{x-\alpha}+\frac{1}{x-\beta}-\frac{1}{x-\delta}-\frac{1}{x-\gamma}\right\}$$

$$\left\{\frac{1}{x-\beta}+\frac{1}{x-\gamma}-\frac{1}{x-\alpha}-\frac{1}{x-\delta}\right\}=0\,, \qquad (3)$$

from which Mr. Russell deduces geometrically the following proposition: "If the roots of a quartic be represented by four points in a plane,* the roots of the sextic covariant are those six points (all real*) from which as origin the quadrilateral inverts into a parallelogram."

Mr. Russell also gives a geometrical interpretation of the H covariant.

49. *Equal roots in the quartic.* If $\alpha=\beta$, then $D=0$, $S=2(\alpha-\gamma)^2(\delta-\alpha)^2$, $T=2(\gamma-\alpha)^3(\alpha-\delta)^3$, and $S^3-2T^2=0$. As indicated in Art. 13, the names D, S, T, etc., refer to the covariant and invariant root functions, which are certain numerical multiples of the corresponding coefficient functions. If D_0, S_0, T_0 represent the coefficient functions as given by Salmon and others, the following relations exist:

$$216D_0=a_0^6(\alpha-\beta)^2(\alpha-\gamma)^2(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^2(\gamma-\delta)^2=a_0^6D\,,$$

$$24S_0=a_0^2\,\Sigma(\alpha-\beta)^2(\gamma-\delta)^2=a_0^2S\,,$$

$$432T_0=a_0^3\,\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta)=a_0^3T\,.$$

Substituting for T and S their values in terms of T_0 and S_0, we have $S_0^3-27T_0^2=0$, when $\alpha=\beta$. Since D_0 also vanishes when $\alpha=\beta$, and is a function of the same order and weight as $S_0^3-27T_0^2$,

$$\therefore\quad S_0^3-27T_0^2=mD_0\,.$$

* Mr. Russell represents all the roots of the quartic and of its covariant, on the complex plane. His use of the word *real* probably refers to the actual existence of the covariant points as determined by a real construction in this plane.

It can be shown by comparison of terms that the numerical multiplier m is equal to unity; and hence that

$$S_0^3 - 27T_0^2 = D_0 .$$

From this, or by an independent comparison of the root functions, we have

$$S^3 - 2T^2 = 64D .$$

50. *I of the quintic and E of the sextic.* The invariant named I, of the quintic, by Hermite,* is made up of five sets of three factors each;* one set of which is as follows:

$$\begin{vmatrix} \alpha^2 & 2\alpha & 1 \\ \beta\gamma & \beta+\gamma & 1 \\ \delta\varepsilon & \delta+\varepsilon & 1 \end{vmatrix} \begin{vmatrix} \alpha^2 & 2\alpha & 1 \\ \beta\delta & \beta+\delta & 1 \\ \gamma\varepsilon & \gamma+\varepsilon & 1 \end{vmatrix} \begin{vmatrix} \alpha^2 & 2\alpha & 1 \\ \beta\varepsilon & \beta+\varepsilon & 1 \\ \gamma\delta & \gamma+\delta & 1 \end{vmatrix} .$$

The vanishing of I is the condition that one root be a double point of the involution formed by the other four; that is, that one root bear the same relation to the other four as do the J points of the quartic to the quartic roots (see Art. 48).

E of the sextic equated to zero expresses the condition that the sextic points should form an involution, since

$$E = \Pi \begin{vmatrix} \alpha\beta & \alpha+\beta & 1 \\ \gamma\delta & \gamma+\delta & 1 \\ \varepsilon\zeta & \varepsilon+\zeta & 1 \end{vmatrix} .$$

51. *Polars.—Covariants of the system of nic and 1ic.* Writing $f(x+\lambda x',\ y+\lambda y')$ in the Joachimstal form

$$f(x+\lambda x', y+\lambda y') = f(x, y) + \lambda\Delta f + \frac{\lambda^2}{2!}\Delta^2 f + \ldots + \frac{\lambda^{n-1}}{(n-1)!}\Delta^{n-1}f + f(x', y'), \quad (1)$$

where

$$\Delta = x'\frac{d}{dx} + y'\frac{d}{dy} .$$

The functions $\Delta^r f(x, y)$ are called the polars of the point $x'y'$ with regard to the binary $n$ic, f; these polars are covariants of the system of the n points of f with the point $x'y'$; they are the emanants† of the $n$ic f. The polars denote groups of points related to the $n$ic points and the point $x'y'$.

* *Camb. and Dublin Math. Journ.*, Vol. IX, p. 186, etc. Salmon, p. 258.

† See Art. 23.

Let $f(x, y)$ be the quartic

$$(\alpha_1 x + \alpha_2 y)(\beta_1 x + \beta_2 y)(\gamma_1 x + \gamma_2 y)(\delta_1 x + \delta_2 y), \tag{2}$$

which, for convenience, may be written

$$(x - \alpha y)(x - \beta y)(x - \gamma y)(x - \delta y), \tag{3}$$

then

$$\varDelta f = \Sigma (x' - \alpha y')(x - \beta y)(x - \gamma y)(x - \delta y), \tag{4}$$

$$= (x' - \alpha y')(x' - \beta y')(x' - \gamma y')(x' - \delta y') \, \Sigma \frac{(x - \beta y)(x - \gamma y)(x - \delta y)}{(x' - \beta y')(x' - \gamma y')(x' - \delta y')}. \tag{5}$$

Writing x for $\dfrac{x}{y}$, and m for $(x' - \alpha)(x' - \beta)(x' - \gamma)(x' - \delta)$,

$$\varDelta f = m \Sigma \frac{(x - \beta)(x - \gamma)(x - \delta)}{(x' - \beta)(x' - \gamma)(x' - \delta)}.* \tag{6}$$

Similarly $\varDelta^2 f = m \Sigma \dfrac{(x - \gamma)(x - \delta)}{(x' - \gamma)(x' - \delta)}$, $\varDelta^3 f = m \Sigma \dfrac{(x - \delta)}{(x' - \delta)}$.

In general, if $f(x, y)$ be an $n$ic, its rth polar is

$$\varDelta^r f(x,y) = [(x' - \alpha)(x' - \beta)(x' - \gamma) \ldots] \, \Sigma \frac{x - \alpha}{x' - \alpha} \cdot \frac{x - \beta}{x' - \beta} \cdot \frac{x - \gamma}{x' - \gamma} \ldots \tag{7}$$

to $(n - r)$ factors.

Since each polar of x' with respect to f is the first polar of x' with respect to the preceding polar, the theory of first polars includes that of all polars. In the binary as in the ternary forms: If A lie on the rth polar of B, then B lies on the $(n - r)$th polar of A; or more accurately, if A be one of the rth polar points of B, then B is one of the $(n - r)$th polar points of A.

52. *The* 1*st and* $(n - 1)$th *polars of the* $n$ic. Equating to zero the first polar,

$$\Sigma \left[\frac{x' - \alpha}{x - \alpha}\right] \tag{1}$$

$$= \Sigma \left[1 - \frac{x' - x}{\alpha - x}\right] = 0; \tag{2}$$

$$\therefore \quad \frac{n}{x' - x} = \Sigma \frac{1}{\alpha - x}; \tag{3}$$

$\therefore$ the harmonic mean of the n points $\alpha, \beta, \gamma, \ldots$, with respect to x as origin,

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* Dr. Bôcher (*Annals of Math.*, Vol. 7, p. 70) remarks that $f'(x)$ is the 1st polar of $f(x)$ with respect to $x' = \infty$; and that by fractional linear transformation we pass from this case to that of the 1st polar with respect to any point P.

is the point x', where x denotes any one of the $(n-1)$ points of the first polar; and the quasi-inverse of the point x' with respect to x, is the arithmetical mean of the quasi-inverse* of $\alpha, \beta, \gamma, \ldots$, with respect to the same point x.

The equation of the $(n-1)$th polar may be written

$$\Sigma \left[\frac{x-\alpha}{x'-\alpha}\right] = \Sigma \left[1 - \frac{x'-x}{x'-\alpha}\right] = 0;$$

$$\therefore \quad \Sigma \frac{1}{\alpha - x'} = \frac{n}{x-x'};$$

and x is the harmonic mean of the n points $\alpha, \beta, \gamma, \ldots$, with respect to x' as origin; i. e. the quasi-inverse of x with respect to x', is the arithmetical mean of the quasi-inverse of $\alpha, \beta, \ldots$, with respect to x'.

Chapter III.

Binary Root Forms in Their Relations to Certain Ternary Forms.†

53. Contravariants and mixed-concomitants have been defined by Salmon as invariant functions into which enter variables λ, μ, ν that are transformed by the inverse transformation.‡ It is proposed in the present chapter to consider certain classes of ternary contravariants derived from binary invariants, and the mixed-concomitants that are derived directly from binary covariants, and also the geometrical interpretations of these and other contravariants and mixed-concomitants. A new use will be made of semi-invariants in their root-forms; and from them will be derived certain semi-contravariants, so named by analogy.

54. *Binary invariants and contravariants.* Let $f(x, y, z) = 0$ be the equation of a curve and $\lambda x + \mu y + \nu z = 0$, of a transversal; then $f\left[x, y, -\frac{\lambda}{\nu}x - \frac{\mu}{\nu}y\right] = 0$ which we shall call $\varphi(x, y) = 0$, is the equation of a system of lines from the point $x = 0$, $y = 0$ to the intersections of the transversal and curve, or, in other words, the equation of a system of points on the transversal. Any given invariant relation§ among the roots of $\varphi(x, y)$, expressed as a relation among the intersections of $f(x, y, z)$ with the line $\lambda x + \mu y + \nu z$, is a contravariant of $f(x, y, z)$ equated to zero, and is the tangential equation of the envelope of the line $\lambda x + \mu y + \nu z = 0$ subject to the given invariant relation among its

* See first note to Art. 44.

† The writer is indebted to Professor McMahon for suggestions on the subject of this chapter.

‡ See M. H. Algebra, pp. 117, 120, 127, 145.

§ By "invariant relation" is to be understood an invariant equated to zero; similarly for "covariant relation."

intersections with $f(x, y, z) = 0$. E. g. the T invariant relation of a quartic is

$$\Sigma(\alpha - \beta)^2 (\gamma - \delta)^2 (\alpha - \gamma)(\beta - \delta) = 0 \tag{1}$$

or

$$\overline{12}^2 . \overline{23}^2 . \overline{31}^2 = 0, \qquad \text{[Cayley's notation.]} \tag{2}$$

which is the condition for a harmonic relation among the roots of the quartic (Art. 47). Expressing these operations, to be performed on the binary quartic $\varphi(x, y)$, in terms of operations* on the ternary quartic $f(x, y, z)$, we get

$$\overline{\lambda 12}^2 . \overline{\lambda 23}^2 . \overline{\lambda 31}^2 . f_1 f_2 f_3 = 0 . \tag{3}$$

The first member of (3) is equivalent to the T invariant of the quartic $f\left[x, y, -\frac{\lambda}{\nu}x - \frac{\mu}{\nu}y\right]$, and may be called the T contravariant of $f(x, y, z)$. Moreover (3) is the tangential equation of the curve of the sixth class, which is the envelope of a transversal (λ, μ, ν) divided harmonically by the quartic curve.

From the above it is evident that to any invariant of a binary $n$ic, there corresponds a contravariant of the ternary $n$ic, which, when equated to zero, represents the envelope of a transversal satisfying the given invariant relation among its intersections with the ternary $n$ic.

55. *Mixed concomitants derived from binary covariants.* Similarly, to every covariant of a binary $n$ic there corresponds a mixed concomitant in $(x, y, z, \lambda, \mu, \nu)$, which, when equated to zero, represents a curve whose intersections on the line (λ, μ, ν) bear the given covariant relation to the intersections of $f(x, y, z) = 0$, on the same transversal. E. g. to the covariant $\overline{12}^2 \varphi_1 \varphi_2$ of a binary cubic corresponds the mixed concomitant $\overline{\lambda 12}^2 f_1 f_2$ of a ternary, which, when equated to zero, represents a conic whose intersections with $\lambda x + \mu y + \nu z = 0$ are the points denoted by $\overline{12}^2 \varphi_1 \varphi_2 = 0$, where the roots of $\varphi = 0$ give the intersections of the cubic curve $f(x, y, z) = 0$ with the same line. If A, B, C be these intersections, then either of the intersections P of the conic satisfies the relation

$$PA^2 . BC^2 + PB^2 . CA^2 + PC^2 . AB^2 = 0 .$$

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* $\overline{12}\, \varphi_1 \varphi_2 = \begin{vmatrix} \frac{d}{dx_1} & \frac{d}{dy_1} \\ \frac{d}{dx_2} & \frac{d}{dy_2} \end{vmatrix} \varphi_1 \varphi_2 = \frac{1}{\nu^2} \begin{vmatrix} \lambda & \mu & \nu \\ \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ \frac{d}{dx_2} & \frac{d}{dy_2} & \frac{d}{dz_2} \end{vmatrix} f_1 f_2 = \frac{1}{\nu^2} . \lambda 12\, f_1 f_2 .$

It will be observed that this mixed concomitant locus is not fixed, but varies with the line (λ, μ, ν). We may obtain from it an important fixed concomitant of the given curve by finding the condition that λ, μ, ν must fulfil, in order that one root of the binary covariant may move on another given fixed curve. This condition is evidently the result of eliminating x, y, z from the equations of this fixed curve, the mixed concomitant, and the line (λ, μ, ν).

56. *Two methods of deriving a new set of envelopes concomitant to a given curve.* The preceding interpretations of contravariants and mixed concomitants are of use in obtaining a set of envelopes by the following methods. The first method is applicable to the derivation of contravariant envelopes from those binary covariants in which the order and degree are equal; the second method has a more general application.

Example of first method. Let it be required to find the envelope of a transversal of a cubic, such that there exists among A, B, C, the three points of intersection, the relation

$$AB \,.\, AC + BC \,.\, BA + CA \,.\, CB = 0,$$

which corresponds to the semi-invariant

$$\Sigma(\alpha-\beta)(\alpha-\gamma)=0\,. \tag{1}$$

The covariant relation of which (1) is the "source" is

$$\Sigma(\alpha-\beta)(\alpha-\gamma)(x-\beta)(x-\gamma)=0\,, \tag{2}$$

which, when (1) is satisfied, has an infinite root. Substituting δ for x in (2), it becomes the S invariant of a quartic; i. e.

$$\Sigma(\alpha-\beta)(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)=0$$

in which δ is infinite. Hence the problem is reduced to the more familiar one exemplified above: To find the envelope of a transversal of a quartic curve such that the S invariant of the four intersections may vanish; the "quartic curve" being in this case a composite, made up of the given cubic and the line at infinity.

Since the Cayley symbol for S of a binary quartic is $\overline{12}^4\varphi_1\varphi_2$, the symbolic equation of the required envelope is

$$\overline{\lambda 12}^4 \,.\, f_1 f_2 = 0$$

where $f(x, y, z)$ is the product of the given ternary cubic by the linear function $ax+by+cz$; the coefficients a, b, c being the lengths of the sides of the triangle of reference.

58. *General statement of first method.* The following is a general statement of the method pursued in the above example. It is required to find the envelope of a transversal which moves so as to satisfy a given semi-invariant relation among its intersections with the $n$ic curve. If in the covariant of which the given invariant is the source, the order and degree be equal, there is an invariant relation of a binary $(n+1)$ic which is consistent with the given semi-invariant relation, if one root of the $(n+1)$ic be infinite. This invariant relation of the binary $(n+1)$ic may be regarded as an invariant relation of the binary $n$ic and the linear $x/y=\infty$; and from it may be derived a corresponding contravariant relation for the ternary $n$ic curve, and the line at infinity; which is the tangential equation of the required envelope of the transversal (λ, μ, ν).

It will be observed that the covariant relation of the binary $n$ic with one root of the covariant infinite, and the invariant relation of the binary $(n+1)$ic which has one root infinite, and the contravariant relation of the ternary $(n+1)$ic composed of the given curve and the line at infinity, are all different expressions for the same condition; it is the last form that can be interpreted as the equation of an envelope in (λ, μ, ν) coordinates.

59. *Number of envelopes of the first method.* It is now evident that for every covariant of a binary $n$ic in which the order and degree are equal, there is a contravariant, which is the envelope of the transversal of a ternary $n$ic curve, whose intersections satisfy the semi-invariant relation obtained by equating to zero the source of the covariant of the binary $n$ic. The symbol of this contravariant is derived from that of the invariant of the binary $(n+1)$ic, whose order and weight are equal respectively to the order and weight of the covariant of the $n$ic; and is obtained by prefixing λ to each group of figures in the invariant symbol; and operates upon the product of the ternary $n$ic and the line at infinity. For every invariant of an $(n+1)$ic there is a covariant of the $n$ic, in which the order and degree are equal, and which is obtained by substituting x for one root in the summation symbol of the invariant of the $(n+1)$ic; thus, to the T of the quartic, which is

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2(\alpha-\gamma)(\beta-\delta),$$

there corresponds a covariant of the cubic, which is

$$\Sigma(\alpha-\beta)^2(\gamma-x)^2(\alpha-\gamma)(\beta-x).$$

Therefore, by the preceding paragraph, to every invariant

$$\Sigma(\alpha-\beta)^{e_{12}}\ldots(\alpha-\nu)^{e_{1n}}(\beta-\nu)^{e_{2n}}\ldots(\mu-\nu)^{e_{mn}}$$

of an $n$ic, there corresponds a function of λ, μ, ν, which when equated to zero

is the envelope of a transversal of an $(n-1)$ic curve, whose intersections satisfy the semi-invariant condition obtained by equating to zero the source of

$$\Sigma(\alpha-\beta)^{e_{12}}\ldots(\alpha-x)^{e_{1n}}(\beta-x)^{e_{1n}}\ldots(\mu-x)^{e_{mn}}.$$

60. *These envelopes are semi-contravariants of the nic.* These functions which are contravariants of the system of *n*ic and line at infinity, may be called *semi-contravariants* of the *n*ic; and when equated to zero, each represents a curve such that any tangent to it meets the *n*ic curve in points which satisfy a particular semi-invariant relation; while a *contravariant* curve of an *n*ic has tangents whose intersections with the *n*ic satisfy an invariant relation. These invariant relations being unchanged by any linear transformation, the contravariant curves of Art. 54 bear relations to the *n*ic curve which are unchanged by any linear transformation. Semi-contravariant curves maintain their relations to the *n*ic curve only under such transformations as preserve the semi-invariant relations of the tangent and the *n*ic to which it is transversal (see Art. 42). These semi-invariant relations are preserved only by such transformations as are equivalent to projection with the centre of projection at infinity, or projection with the axis of projection at infinity.

61. *Applications of the first method.* (a). To find the envelope of the transversal of a quintic curve such that the intersections A, B, C, D, E may satisfy the relation $\Sigma BC^2 . DA^2 = 0$: This corresponds to the semi-invariant

$$\Sigma(\beta-\gamma)^2(\delta-\alpha)^2=0,$$

which is the source of the covariant

$$\Sigma(\beta-\gamma)^2(\delta-\alpha)^2(x-\varepsilon)^2=0.$$

Hence the latter must have an infinite root. Changing x into ζ, we obtain the invariant of a sextic,

$$\Sigma(\beta-\gamma)^2(\delta-\alpha)^2(\zeta-\varepsilon)^2=0,$$

one of the roots of the sextic being infinite. The symbolical form of this invariant is

$$\overline{12}^6\varphi_1\varphi_2=0,$$

which gives rise as before to the ternary contravariant

$$\overline{\lambda 12}^6 f_1 f_2=0.$$

Taking f as the product of the given quintic curve and the line at infinity, this operation will give the tangential equation of the required envelope.

(b). Similarly a transversal of a septic curve satisfying the relation $\Sigma AB^2 \,.\, CD^2 \,.\, EF^2 = 0$ has the envelope

$$\overline{\lambda 12}^8 f_1 f_2 = 0\,,$$

in which f is, as before, a composite octic; and so in general for any curve of odd degree, when the expression $\Sigma AB^2 \,.\, CD^2 \ldots$ involves all the points but two, in separate pairs.

(c). Starting with the invariant of a quartic $(\overline{12} \,.\, \overline{23} \,.\, \overline{13})^2\, \varphi_1\varphi_2\varphi_3$, we know that there is a contravariant of a quartic curve,

$$(\overline{\lambda 12} \,.\, \overline{\lambda 23} \,.\, \overline{\lambda 13})^2 f_1 f_2 f_3 = 0\,.$$

Applying this to the composite quartic made up of a cubic curve and line at infinity, and thus deriving a semi-contravariant of the cubic, let us enquire what geometrical property this envelope possesses.

The root-form of the above invariant is found to be

$$\Sigma(\alpha - \beta)^2 (\alpha - \gamma)(\delta - \gamma)^2 (\delta - \beta)\,,$$

and the geometrical property of the corresponding contravariant of a quartic curve is that its tangents satisfy the relation $\Sigma AB^2 \,.\, AC \,.\, DC^2 \,.\, DB = 0$. Letting one of these points D move on the line at infinity, the other three must satisfy the relation

$$\Sigma AB^2 \,.\, AC = 0\,,$$

which is the geometrical property of the above semi-contravariant of a cubic curve.

(d). Similarly, from the invariant $(\overline{12} \,.\, \overline{23} \,.\, \overline{31})^4\, \varphi_1\varphi_2\varphi_3$ of a binary octic is derived the semi-contravariant $(\overline{\lambda 12} \,.\, \overline{\lambda 23} \,.\, \overline{\lambda 31})^4 f_1 f_2 f_3$ of a ternary septic, where f is the composite ternary octic. The corresponding semi-invariant relation is found to be

$$\Sigma AB^2 \,.\, CD^2 \,.\, EF^2 \,.\, AC \,.\, BD \,.\, EG = 0\,.$$

There is a similar semi-contravariant for any ternary $(4r - 1)$ic.

62. *Second method of deriving semi-contravariants of ternaries from semi-invariant root forms of binaries.* If in the covariant of which the given semi-invariant is the source, the order and degree be not equal, there is evidently no corresponding invariant of the binary $(n + 1)$ic, and therefore no corresponding contravariant of the ternary $(n + 1)$ic. In this case, in order to find an envelope, corresponding to the given semi-invariant condition, we make use of the mixed concomitant of Art. 55. E. g. let it be required to find

the envelope of the transversal of a quartic, that moves so that its intersections A, B, C, D satisfy $\Sigma AB^2 = 0$, or

$$\Sigma(\alpha - \beta)^2 = 0. \tag{1}$$

The covariant relation

$$\Sigma(\alpha - \beta)^2 (x - \gamma)^2 (x - \delta)^2 = 0 \tag{2}$$

must then have a value of x infinite, and the corresponding mixed concomitant

$$\overline{\lambda 12}^2 f_1 f_2 = 0\,, \tag{3}$$

where f is a ternary quartic, must have one intersection with

$$\lambda x + \mu y + \nu z = 0\,, \tag{4}$$

at infinity ; i. e. (3) and (4) must intersect on

$$ax + by + cz = 0\,, \tag{5}$$

where a, b, c are the lengths of the sides of the reference triangle. The eliminant of (3), (4), (5) is obtained by substituting for x, y, z in (3), the expressions $c\mu - b\nu$, $a\nu - c\lambda$, $b\lambda - a\mu$. The result is the condition the transversal (λ, μ, ν) must fulfil, in order that its intersections with the curve $f(x, y, z) = 0$, may satisfy the relation (1).

63. *The envelopes of the second method include those of the first method.* The envelope of a transversal satisfying any given semi-invariant relation can be found by the second method; since to every semi-invariant of an $n$ic, there corresponds a covariant of the same $n$ic, and to the covariant a mixed concomitant of the ternary $n$ic. Hence the envelopes whose equations are obtainable by the second method, include those that may be found by the first method. E. g. the envelope of the transversal of a cubic, moving so that its intersections satisfy $\Sigma AB\,.\,AC = 0$, which has been shown in Art. 57 to be

$$\overline{\lambda 12}^4 f_1' f_2' = 0, \qquad (f' \equiv \text{a composite ternary quartic.})$$

is also by Art. 62,

$$\overline{\lambda 12}^2 f_1 f_2 = 0\,, \qquad (f \equiv \text{the ternary cubic.})$$

in which x, y, z are to be replaced by $b\nu - c\mu$, $c\lambda - a\nu$, $a\mu - b\lambda$.

It may be of interest to compare the actual working out of these indicated operations, for the canonical form of the cubic, viz:

$$f \equiv x^3 + y^3 + z^3 + 6mxyz\,,$$

$$f' \equiv (ax + by + cz) f$$

$$\equiv ax^4 + bx^3y + axy^3 + by^4 + z(cx^3 + 6amx^2y + 6bmxy^2 + cy^3)$$

$$+ z^2(6cmxy) + z^3(ax + by) + cz^4\,.$$

Writing ξ, η, ζ for $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, then the operator of the second method is

$$\overline{\lambda 12}^2 = \begin{vmatrix} \lambda & \mu & \nu \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{vmatrix}^2,$$

in which the coefficient of λ^2 is $(\eta_1\zeta_2 - \eta_2\zeta_1)^2$, i. e. $\eta_1^2 . \zeta_2^2 + \zeta_1^2 . \eta_2^2 - 2\eta_1\zeta_1 . \eta_2\zeta_2$. Operating with this on $f_1 . f_2$ and making the subscripts identical gives

$$2\left[\frac{d^2f}{dy^2} . \frac{d^2f}{dz^2} - \left[\frac{d^2f}{dydz}\right]^2\right].$$

The coefficient of $2\lambda\mu$ is $(\eta_1\zeta_2 - \zeta_1\eta_2)(\zeta_1\xi_2 - \xi_1\zeta_2)$, i. e. $\eta_1\zeta_1 . \xi_2\zeta_2 - \xi_1\eta_1 . \zeta_2^2 - \zeta_1^2 . \xi_2\eta_2 + \eta_2\zeta_2 . \xi_1\zeta_1$, which, operating on $f_1 . f_2$, gives, when the subscripts are made identical,

$$2\left[\frac{d^2f}{dydz} . \frac{d^2f}{dxdz} - \frac{d^2f}{dxdy} . \frac{d^2f}{dz^2}\right].$$

Hence the equation of the mixed concomitant curve is

$$\lambda^2 (yz - m^2x^2) + \mu^2 (zx - m^2y^2) + \nu^2 (xy - m^2z^2)$$
$$+ 2\lambda\mu (m^2xy - mz^2) + \ldots = 0.$$

The condition that one of its intersections with (λ, μ, ν) may be on the line at infinity is obtained by putting $b\nu - c\mu$ for x, etc.; and the tangential equation of the required envelope reduces to

$$bc\lambda^4 + (2ma^2 - bc)(\lambda\mu^3 + \lambda\nu^3) + 4(m^2c^2 - mab)\lambda^2\mu^2$$
$$+ (a^2 - 4m^2bc)\lambda^2\mu\nu + \ldots = 0.$$

On the other hand the operator of the first method is

$$\overline{\lambda 12}^4 = \begin{vmatrix} \lambda & \mu & \nu \\ \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \end{vmatrix}^4,$$

in which the coefficient of λ^4 is $(\eta_1\zeta_2 - \eta_2\zeta_1)^4$, i. e. $\eta_1^4 . \zeta_2^4 - 4\eta_1^3\zeta_1 . \eta_2\zeta_2^3 + 6\eta_1^2\zeta_1^2 . \eta_2^2\zeta_2^2 - \ldots$, which operating on $f_1' . f_2'$, gives

$$2\left[\frac{d^4f'}{dy^4} . \frac{d^4f'}{dz^4} - 4\frac{d^4f'}{dy^3dz} . \frac{d^4f'}{dydz^3} + 3\left[\frac{d^4f'}{dy^2dz^2}\right]^2\right]$$
$$= 2[24 . 24bc - 4 . 6 . 6bc] = 864bc.$$

The coefficient of $\lambda^3\mu$ is $4(\eta_1\zeta_2 - \zeta_1\eta_2)^3(\zeta_1\xi_2 - \xi_1\zeta_2)$, which gives

$$-8\left[\frac{d^4f'}{dxdy^3}\cdot\frac{d^4f'}{dz^4} - 3\frac{d^4f'}{dxdy^2dz}\cdot\frac{d^4f'}{dydz^3} + 3\frac{d^4f'}{dxdydz^2}\cdot\frac{d^4f'}{dy^2dz^2} - \frac{d^4f'}{dxdz^3}\cdot\frac{d^4f'}{dy^3dz}\right]$$

$$= -8(6\,.\,24ac - 3\,.\,12\,.\,6b^2m - 6\,.\,6ac) = 864\,(2mb^2 - ac)\,.$$

The coefficients of $\lambda^2\mu^2$, and of $\lambda^2\mu\nu$, may be similarly obtained, and it will be seen that the result agrees with that given above.

The first method (the last worked out above) has the advantage of giving the result directly in terms of λ, μ, ν, without the substitution for x, y, z; while the second method (the first worked out above) has the greater advantage of a simpler differential operator.

64. *Further applications of the second method.* (1) A transversal to a cubic has the two intercepts equal; find its envelope.

Taking $AB = BC$, then $\beta - \alpha = \gamma - \beta$, i. e. $2\beta - \gamma - \alpha = 0$. Similarly, if $BC = CA$, then $2\gamma - \alpha - \beta = 0$; and if $CA = AB$, then $2\alpha - \beta - \gamma = 0$. In any case the symmetric relation

$$(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta) = 0\,,$$

which is semi-invariant, is satisfied. It may also be written

$$\Sigma(\alpha - \beta)^2(\alpha - \gamma) = 0\,,$$

and it is the source of the covariant equation

$$\Sigma(\alpha - \beta)^2(\alpha - \gamma)(x - \beta)(x - \gamma)^2 = 0\,,$$

of which the Cayley symbol is

$$\overline{12}^2\,.\,\overline{13}\,\varphi_1\varphi_2\varphi_3 = 0\,,$$

and the corresponding mixed concomitant is

$$\lambda\overline{12}^2\,\lambda\overline{13}\,f_1f_2f_3 = 0\,. \qquad [f \text{ a ternary cubic.}$$

The coefficient of λ^3 is $(\eta_1\zeta_2 - \eta_2\zeta_1)^2(\eta_1\zeta_3 - \zeta_1\eta_3)$, i. e.

$$\eta_1^{\,3}\,.\,\zeta_2^{\,2}\,.\,\zeta_3 - \zeta_1^{\,3}\,.\,\eta_2^{\,2}\,.\,\eta_3 + \eta_1\zeta_1^{\,2}\,.\,\eta_2^{\,2}\,.\,\zeta_3 + \ldots$$

which, operating on the canonical form of the cubic, gives

$$6\,.\,6z(3z^2 + 6mxy) - 6\,.\,6y(3y^2 + 6mxz) = 108(z^3 - y^3)\,.$$

The coefficient of $\lambda^2\mu$ is

$$(\eta_1\zeta_2 - \zeta_1\eta_2)^2(\zeta_1\xi_3 - \xi_1\zeta_3) + 2(\eta_1\zeta_2 - \zeta_1\eta_2)(\zeta_1\xi_2 - \xi_1\zeta_2)(\eta_1\zeta_3 - \zeta_1\eta_3)\,,$$

i. e.

$$\zeta_1^3 \cdot \eta_2^2 \cdot \xi_3 + 4\xi_1\eta_1\zeta_1 \cdot \eta_2\zeta_2 \cdot \zeta_3 + 2\zeta_1^3 \cdot \xi_2\eta_2 \cdot \eta_3 + 2\xi_1\eta_1\zeta_1 \cdot \zeta_2^2 \cdot \eta_3 + \ldots ,$$

which gives

$$108\,[(1 + 8m^3)\,x^2y + 6mz\,(y^2 + 2mz)]\,.$$

The coefficient of $\lambda\mu\nu$ will be found to vanish.

Hence the mixed concomitant curve is

$$\lambda^3(z^3 - y^3) + \mu^3(x^3 - z^3) + \nu^3(y^3 - x^3) + \lambda^2\mu\,[(1 + 8m^3)\,x^2y + 6mz\,(y^2 + 2mxz)]$$
$$- \lambda\mu^2\,[(1 + 8m^3)\,xy^2 + 6mz\,(y^2 + 2mxz)] + \ldots = 0\,.$$

The semi-contravariant is obtained by putting $c\mu - b\nu$ for x, etc.

An interesting special case is when m is infinite, the cubic reducing to $xyz = 0$, and the mixed concomitant to

$$\lambda^2\mu x^2y - \lambda\mu^2xy^2 + \mu^2\nu y^2z - \mu\nu^2yz^2 + \nu^2\lambda z^2x - \nu\lambda^2zx^2 = 0$$

i. e.

$$(\lambda x - \mu y)\,(\mu y - \nu z)\,(\nu z - \lambda x) = 0\,;$$

and the semi-contravariant breaks up into three second class envelopes $a\mu\nu + b\nu\lambda - 2c\lambda\mu = 0$, etc., which are evidently touched by the four lines whose coordinates are (1, 0, 0), (0, 1, 0), (0, 0, 1), (a, b, c), and hence are parabolas touching the sides of the given triangle. If the line at infinity be projected into the field, the above is the envelope of a line divided harmonically by four given lines; the three conics corresponding to the three ways of pairing the four lines.

(2) To find the envelope of a transversal to a quartic curve, when two non-adjacent intercepts are equal.

The relation $(\beta + \gamma - \alpha - \delta)(\gamma + \alpha - \beta - \delta)(\alpha + \beta - \gamma - \delta) = 0$ may be written $\Sigma(\alpha - \beta)^2(\alpha - \gamma) = 0$, and is the source of $\overline{12}^2 . \overline{13}\,\varphi_1\varphi_2\varphi_3 = 0$. Hence, as before, the required envelope is the eliminant of the equations

$$\left.\begin{aligned} \lambda\overline{12}^2 . \lambda\overline{13}\,f_1 f_2 f_3 &= 0 \\ \lambda x + \mu y + \nu z &= 0 \\ ax + by + cz &= 0 \end{aligned}\right\} \qquad (1)$$

As a special case, let the quartic consist of the four concurrent lines

$$f \equiv xy\,(x^2 - y^2) = 0\,,$$

then

$$\lambda\overline{12}^2 . \lambda\overline{13}\, f_1 f_2 f_3 = \nu^3 (\xi_1\eta_2 - \eta_1\xi_2)^2 (\xi_1\eta_3 - \eta_1\xi_3)\, f_1 f_2 f_3$$

$$= \nu^3 [(\xi_1^3\eta_2^2 + \xi_1\eta_1^2 . \xi_2^2 - 2\xi_1^2\eta_1 . \xi_2\eta_2)\eta_3 - (\xi_1^2\eta_1 . \eta_2^2 + \eta_1^3\xi_2^2 - 2\xi_1\eta_1^2 . \xi_2\eta_2)\xi_3]\, f_1 f_2 f_3$$

$$= \nu^3 [(-36xy^2 - 36x^3)(x^3 - 3x^2y) + (36x^2y + 36y^3)(3x^2y - y^3)]$$

$$= -36\nu^3 (x^2 + y^2)(x^4 - 6x^2y^2 + y^4)$$

$$= -36\nu^3 [x^2 + y^2][x + (\sqrt{2} + 1)y][x - (\sqrt{2} + 1)y][x + (\sqrt{2} - 1)y]$$

$$[x - (\sqrt{2} - 1)y] . \quad (2)$$

When the elimination indicated above is performed, it will be seen that the required envelope breaks up into nine points, three of which coincide at the origin, and six are on the line at infinity, in the directions given by equating to zero the respective factors in (2); and transversals parallel to any of these six directions have two non-adjacent intercepts equal; four of these directions being real, and two imaginary.

(3). To find the envelope of a transversal to an *n*ic curve, such that

$$\Sigma AB^2 . CD^2 . AE = 0 .$$

Since $\Sigma(\alpha - \beta)^2 (\gamma - \delta)^2 (\alpha - \varepsilon)$ is the source of the covariant $\overline{12}^4 . \overline{13}\, \varphi_1\varphi_2\varphi_3$, the required condition is the eliminant of the equations

$$\lambda\overline{12}^4 . \lambda\overline{13}\, f_1 f_2 f_3 = 0 ,$$

$$\lambda x + \mu y + \nu z = 0 ,$$

$$ax + by + cz = 0 .$$

65. *Semi-contravariants.* All the envelopes obtained by the two methods of Arts. 57, 62 are called semi-contravariants because they preserve their relations to the given *n*ic curve only under such linear transformations as preserve the given semi-invariant relation among the segments of a transversal. These partial linear transformations have the same effect as a projection that does not alter the mutual ratio of the segments of a line. This may be a conical projection of a figure from a plane to a parallel plane, or a cylindrical projection to any plane whatever. In the former case the axis of projection is at infinity, and in the latter the centre of projection is an infinitely distant point. In both cases the line at infinity in the first plane projects into the line at infinity in the second plane.

66. *Contravariants of the system of a curve and line.* These semi-contravariants of a curve are contravariants of the system of curve and line at

infinity; for if the line $ax + by + cz = 0$ be replaced by the general line $\lambda' x + \mu' y + \nu' z = 0$, the result of the elimination is the condition that a root of a certain covariant of the binary quantic that gives the intersections of the transversal and curve, may lie on the line (λ', μ', ν'). This relation is not altered if the whole system be subjected to any linear transformation, or to any conical projection; hence the envelope of the transversal is a contravariant curve of the system of $n$ic and line (λ', μ', ν').

On the other hand if the line (λ, μ, ν) be fixed, and (λ', μ', ν') variable, the contravariant equated to zero gives the condition that the line (λ', μ', ν') may pass through some one of the fixed points on (λ, μ, ν) that represent the roots of the binary covariant, and must therefore be the tangential equation of this system of points. In this point of view the function in question is a contravariant of the system of curve and transversal (λ, μ, ν).

It may also be regarded as an invariant of the system composed of the curve and two lines.

From these contravariants new invariants may be derived by replacing the coordinates of one or both lines by differential operators.*

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The tables referred to in Part I, Chapter III, will be given in a future number of the ANNALS.

THE RATIONAL FUNCTIONS OF THE CUBIC.

By Mr. Albert M. Sawin, Evanston, Ill.

The more common solutions of the cubic are those where a supposition is made to exist between auxiliary variables introduced in order to effect its reduction. It is the object of this paper to give the solution without this expedient, depending simply upon the relations the roots sustain to the coefficients and for another reason, much more important, to give the solution of the cubic in precisely the same manner as the quadratic is solved. In order that the work may be progressive and connected, the quadratic

$$x^2 + Ax + B = 0\,,$$

whose roots are $x = a$, $x = b$, and whose equation is

$$x^2 - (a + b)x + ab = 0\,,$$

will be solved by the method that, if not the briefest, I still maintain is the simplest and most natural of any method known, and will be introductory to the method given for the cubic.

Equating, we have

$$a + b = -A\,,$$

$$ab = B\,.$$

Squaring the first and subtracting four times the second from it, we obtain

$$a^2 - 2ab + b^2 = A^2 - 4B\,,$$

whence, extracting the square root

$$a - b = \pm\sqrt{A^2 - 4B}\,.$$

Combining this equation with the one above by addition and subtraction, we have

$$a = -\frac{A}{2} + \sqrt{\frac{1}{4}A^2 - 41}\,,$$

$$b = -\frac{A}{2} - \sqrt{\frac{1}{4}A^2 - B}\,;$$

and these being the values of a and b, are the values of x by hypothesis, and so are the roots of the quadratic equation.

Assume the cubic in the form

$$x^3 + Ax + B = 0.$$

Since the sum of the roots is zero, one is the negative sum of the other two, and we have the roots expressed as $x = a$, $x = b$, $x = -(a + b)$. Whence

$$x - a = 0, \quad x - b = 0, \quad x + a + b = 0,$$

whose cubic equation is

$$x^3 - (a^2 + ab + b^2)x + ab(a + b) = 0.$$

Whence, equating with the equation above,

$$-\frac{1}{3}(a^2 + ab + b^2) = \frac{A}{3}, \tag{1}$$

$$\frac{1}{2}(a + b)ab = \frac{B}{2}. \tag{2}$$

Cubing the first,

$$-\frac{1}{27}a^6 - \frac{1}{9}a^5b - \frac{2}{9}a^4b^2 - \frac{7}{27}a^3b^3 - \frac{2}{9}a^2b^4 - \frac{1}{9}ab^5 - \frac{1}{27}b^6 = \frac{A^3}{27}.$$

Squaring the second,

$$\frac{1}{4}a^4b^2 + \frac{1}{2}a^3b^3 + \frac{1}{4}a^2b^4 = \frac{B^2}{4}.$$

Adding these equations we obtain

$$-\frac{1}{27}a^6 - \frac{1}{9}a^5b + \frac{1}{36}a^4b^2 + \frac{13}{54}a^3b^3 + \frac{1}{36}a^2b^4 - \frac{1}{9}ab^5 - \frac{1}{27}b^6 = \frac{A^3}{27} + \frac{B^2}{4}.$$

Factoring from the first member $-\frac{3}{64}$,

$$-\frac{3}{64}\left[\frac{64}{81}a^6 + \frac{64}{27}a^5b - \frac{64}{108}a^4b^2 - \frac{832}{162}a^3b^3 - \frac{64}{108}a^2b^4 + \frac{64}{27}ab^5 + \frac{64}{81}b^6\right] = \frac{A^3}{27} + \frac{B^2}{4}.$$

Extracting the square root of this perfect square,

$$\frac{1}{8}\sqrt{-3}\left[\frac{8}{9}a^3 + \frac{4}{3}a^2b - \frac{4}{3}ab^2 - \frac{8}{9}b^3\right] = \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}. \tag{3}$$

Adding $\frac{1}{8}(a^3 + b^3)$ to equation (2), and also subtracting the same, we have

$$-\frac{1}{8}(a^3 + 3a^2b + 3ab^2 + b^3 - a^3 + a^2b + ab^2 - b^3) = -\frac{B}{2}. \tag{4}$$

Adding equations (4) and (3) we have

$$-\frac{1}{8}\Big[a^3 + 3a^2b + 3ab^2 + b^3 - (a^3 + a^2b - ab^2 - b^3)\sqrt{-3} - a^3 + a^2b + ab^2 - b^3$$
$$+ \left\{\frac{1}{9}a^3 - \frac{1}{3}a^2b + \frac{1}{3}ab^2 - \frac{1}{9}b^3\right\}\sqrt{-3}\Big] = -\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}.$$

Extracting the cube root of this perfect cube,

$$-\frac{a+b}{2} + \frac{a-b}{2\sqrt{-3}} = \sqrt[3]{-\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}. \qquad (5)$$

Subtracting (4) from (3),

$$-\frac{1}{8}\Big[a^3 + 3a^2b + 3ab^2 + b^3 + (a^3 + a^2b - ab^2 - ab^3)\sqrt{-3} - a^3 + a^2b + ab^2 - b^3$$
$$- \left\{\frac{1}{9}a^3 - \frac{1}{3}a^2b + \frac{1}{3}ab^2 - \frac{1}{9}b^3\right\}\sqrt{-3} = -\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}.$$

Extracting the cube root of this perfect cube,

$$-\frac{a+b}{2} - \frac{a-b}{2\sqrt{-3}} = \sqrt[3]{-\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}. \qquad (6)$$

Adding (5) and (6),

$$-(a+b) = \sqrt[3]{-\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}} + \sqrt[3]{-\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}. \qquad (7)$$

Subtracting (6) from (5),

$$a - b = \sqrt{-3}\,\sqrt[3]{-\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}} - \sqrt{-3}\,\sqrt[3]{-\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}.$$

Whence, by comparison of these two equations,

$$a = \left\{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}\right\}\sqrt[3]{-\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}$$
$$+ \left\{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right\}\sqrt[3]{-\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}, \qquad (8)$$

$$b = \left\{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}\right\}\sqrt[3]{-\frac{B}{2} + \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}$$
$$+ \left\{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}\right\}\sqrt[3]{-\frac{B}{2} - \sqrt{\frac{A^3}{27} + \frac{B^2}{4}}}. \qquad (9)$$

Thus we arrive at the values of $-(a+b)$, a, and b, equations (7), (8) and (9), and these being the three values of x are the three roots of the cubic equation. The solution, it will be observed, is the analogue of the solution of the quadratic given above. The quartic is readily reduced in the same manner.

Now several important properties of the formula for the cubic may be pointed out, which, so far as I have observed, have hitherto not been given.

The three elements of the formula are obviously

$$\sqrt{\frac{A^3}{27}+\frac{B^2}{4}},\ \sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}},\ \text{and}\ \sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}.$$

It may now be asked what are the rational functions of the roots of the cubic that correspond to each one of these elements. Those who have perused Wantzel's Abstract of Abel's Argument (Serret's Algébre Superieure, Vol. II, p. 393) are aware that he assumed that such rational functions existed theoretically. For the first of the above forms we have, at once, from equation (3),

$$a^3+\frac{3}{2}a^2b-\frac{3}{2}ab^2-b^3=\sqrt{-27\left\{\frac{A^3}{27}+\frac{B^2}{4}\right\}}. \tag{10}$$

Factoring the first member,

$$(a-b)\left\{a+\frac{1}{2}b\right\}(a+2b)=\sqrt{-A^3-\frac{27}{4}B^2}.$$

Or the rational functions of the roots of the cubic corresponding to the *square root radical* of the formula in the product of three linear factors in an arithmetical progression of which the first is the difference of the independent roots, and the common difference in $\frac{3}{2}b$, one of those roots.

It is here useful to observe that in the quadratic the radical corresponds simply to the difference of the roots, which in the cubic is the first factor.

For the second and third elements we have from (5) and (6), directly,

$$-\frac{a+b}{2}+\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}+\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}, \tag{11}$$

and

$$-\frac{a+b}{2}-\frac{a-b}{2\sqrt{-3}}=\sqrt[3]{-\frac{B}{2}-\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}}. \tag{12}$$

These functions are closely associated with the imaginary cubic roots of unity, viz., $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, but not as factors. In (10)

if we substitute any true roots, as a and b, the radical changes sign, but in (11) and (12) the cube root radicals simply interchange places. The cyclic substitutions of the roots all at a time gives in their order the formulas for all the roots the elements remaining unchanged.

In the equations (11) and (12) the reducible case requires that the imaginary term should vanish. The analysis of the term $\frac{a-b}{2\sqrt{-3}}$ with this end in view postulates the conditions which must subsist between the roots in the reducible case, which is obviously that two roots be equal, or that they be conjugate imaginaries, in which case the imaginary factor disappears by division.

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